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The aspect of 'Robustness' for linear multivariable systems in time domain is the central theme of the research under the present contract. Upper bounds on the linear, structured, time varying perturbation of an asymptotically stable linear time invariant regulator are obtained to maintain both stability and acceptable regulation, using Lyapunov approach. Improvement of the proposed measures over existing measures is illustrated with the help of examples. It is shown that by employing a scaling transformation on the nominal system, it is possible to further improve the upper bound. The proposed 'Perturbation Bound Analysis' is used to design robust controllers for Linear Quadratic Regulators with structured uncertainty. Introducing quantitative measures called 'Stability Robustness Index' and 'Performance Robustness Index', design algorithms are presented by which one can achieve a trade off between nominal performance, stability robustness and performance robustness. Applications considered include aircraft control problems, large space structure control problems having uncertain modal data and mode truncation as the perturbations.

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TIME DOMAIN DESIGN OF ROBUST CONTROLLERS FOR LQG REGULATORS; APPLICATION TO LARGE SPACE STRUCTURES

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Foreword

This report was prepared by the Department of Mechanical Engineering, Stevens Institute of Technology, New Jersey 07030. under Air Force Contract F 33615-84-K-3606. The work was performed under the direction of Dr. S. S. Banda Lt. D. B. Ridgely and Dr. V. Venkayya of the Air Force Wright Aeronautical Laboratories, Flight Dynamics Laboratory, Wright Patterson Air Force Base, Ohio 45433.

The technical work was conducted by Dr. R. K. Yedavalli, Principal Investigator and graduate research assistant Mr. Z. Liang. Contract work was performed during the period May 1984 - Aug. 1985.

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TABLE OF CONTENTS

Page

NOMENCLATURE

R^{α}	= real vector space of dimension α
$\rightarrow$	= belongs to
$\lambda[\cdot]$	= Eigenvalues of the matrix $[\cdot]$
$\sigma[\cdot]$	= singular value of the matrix $[\cdot]$
	= $\{\lambda([\cdot][\cdot]^T)\}^{1/2}$
$[\cdot]_s$	= symmetric part of a matrix $[\cdot]$
$\|(\cdot)\|$	= modulus of the entry $(\cdot)$
$[\cdot]_m$	= modulus matrix = matrix with modulus entries
V	= for all
I_{α}	= $\alpha \times \alpha$ Identity matrix
IJC	= International Journal of Control
IEEE AC	= IEEE Transactions on Automatic Control

I. INTRODUCTION AND PERSPECTIVE

It is well known that the inaccuracies in the mathematical models of physical systems can severely compromise the resulting control designs. The errors associated with mathematical models of physical systems may be broadly categorized as i) parameter errors, ii) truncated models (errors in model order), iii) neglected or incorrectly modeled external distrubances and iv) neglected nonlinearities. It is the inevitable presence of these errors in the model used for design that eventually limits the performance attainable from the control system designs produced by either classical (frequency domain) or modern (time domain) control theory. The problem of model errors is more critical, in general, for large scale Linear Quadratic Gaussian (LQG) optimal control problems and in particular, for aerospace applications like Large Space Structure (LSS) control and other aeroelastic systems. These applications are, of course, of extreme importance to the U.S. Air Force. One fundamental problem of LSS control is the control of a large dimensional system with a controller of much smaller dimension (model/controller truncation) compounded with modal data uncertainty (parameter errors). In the light of these observations, it is evident that 'robustness' is an extremely desirable (sometimes, necessary) feature of any feedback control design proposed for LSS control. 'Robustness' studies of LQG regulators is the central theme, of the present research.

For our present purposes a 'robust' control design is that design which behaves in an 'acceptable' fashion (i.e. satisfactorily meets the system

specifications) even in the presence of modeling errors. Since the system specifications could be in terms of stability and/or performance (regulation, time response, etc.) we can conceive two types of robustness, namely, 'Stability Robustness' and 'Performance Robustness'. Limiting our attention in this research to 'parameter errors' and 'mode truncation' as the two types of modeling errors that may cause instability (or performance degradation) in the system, we formally define 'stability robustness' and 'performance robustness' as follows:

'Stability Robustness': Maintaining closed loop system stability in the presence of modeling errors mainly parameter variations and model/controller truncation.

'Performance Robustness': Maintaining satisfactory level of performance (or regulation) in the presence of modeling errors mainly parameter variations and mode truncation.

Clearly, the definition of 'Performance (or Regulation) Robustness' embodies the definition of 'Stability Robustness'. If the main concern is for stability alone, then one can analyze and synthetize for robust stability whereas if both stability and performance (regulation, for the case of linear regulators) are of concern, then one needs to resort to the 'Performance Robustness' aspect. In this report these two aspects are clearly delineated and accordingly are given separate treatment.

The published literature on the 'robustness' of linear systems can be viewed mainly from two perspectives, namely i) frequency domain analysis and ii) time domain analysis. The main direction of research in frequency domain has been to extend and generalize the well known classical single input single output treatment to the case of multiple input multiple output systems,

using the singular value decomposition. In the case of frequency domain results, the perturbations are mainly viewed in terms of 'gain' and 'phase' changes. The time domain treatment is more or less similar to the frequency domain treatment in spirit but quite different in detail. The time domain treatment is more amenable to treating perturbations in the form of parameter variations, nonlinearities and external disturbances and also for the physical interpretation of many real life perturbations. This research treats the robustness analysis and design from time domain viewpoint and in particular focuses on the well known Linear Quadratic Regulator problem. In addition, the main tool used is the Lyapunov stability analysis which allows time varying perturbations to be considered in the analysis.

The problem of maintaining the stability of a nominally stable system subject to perturbations has been an active topic of research for quite some time. One factor which clearly influences this type of analysis is the characterization or type of 'perturbation'. Even in the context of nominally stable linear systems, the 'perturbations' can take different forms like linear, nonlinear, time invariant, time varying, structured and unstructured. Structured perturbations are those for which bounds on the individual elements of the perturbation matrix are known (or derived) whereas unstructured perturbations are those for which only a norm bound on the perturbation matrix is known (or derived). In this research, we focus our attention on linear, time varying, structured perturbations as affecting a nominally stable linear time invariant system.

With this perspective in mind, the report is organized as follows: Section II presents the analysis for stability robustness of linear systems. After a brief literature search, upper bounds on the linear, time varying,

structured perturbation of an asymptotically stable linear time invariant system are presented to maintain stability. Then a state transformation technique is presented to further reduce the conservatism of the Lyapunov stability bound. The analysis is then applied to ultimate boundedness control, the analysis of which also depends on Lyapunov stability bound. The usefulness of the perturbation bound for structured uncertainty is demonstrated by extending the analysis to present sufficient conditions for the stability of an interval matrix. Section III is completely devoted to the design of controllers for robust stability along with an aircraft control example. Then in Section IV attention is focused on the performance robustness analysis, which presents upper bounds for robust regulation and their relationship to the bounds for robust stability. Section V presents a procedure for designing controllers for robust regulation along with a simple example to illustrate the procedure. Section VI addresses the application of this 'Perturbation Bound Analysis' to Large Space Structures (LSS) models in which uncertain modal data and mode truncation are modeled as additive perturbations and the results, obtained by exploiting the special structure of LSS models, are discussed. Finally Section VII offers some concluding remarks and explores avenues for future research that needs the continued sponsorship of the Air Force.

II. ANALYSIS FOR STABILITY ROBUSTNESS OF LINAER SYSTEMS

In the present day applications of linear systems theory and practice, one of the challenges the designer is faced with is, to be able to guarantee 'acceptable' behavior of the system even in the presence of perturbations. The fundamental 'acceptable' behavior of any control design for linear systems is 'stability' and accordingly one of the important tasks of the designer is to assure 'stability' of the system subject to perturbations.

In particular, as discussed in the introduction, we concentrate on 'parameter uncertainty' as the type of perturbation acting on the system. This section, thus, addresses the analysis of 'stability robustness' of linear systems subject to parameter uncertainty.

2.1 Brief Review of Literature

The aspect of robust control design for linear regulators subject to parameter uncertainty has also been an active topic of research in recent years. Much of this research in frequency domain is carried out using singular value analysis. The special issue [1], [2] contain valuable references in this category. A recent contribution in the frequency domain is the work of Daniels and Kouvaritakis [3] who consider 'stability margin' both from tolerance to uncertainty as well as typical gain, phase margin points of view. On the other hand, there is equally interesting research reported in time domain. Davison [4] focuses on the structure or characterization of the robust controller and presents necessary and sufficient conditions for such an observer based controller to exist. It is assumed that the system is stable

under the given range of parameter perturbations and no explicit bounds on the tolerable perturbations for stability are given. The parameter space design of Ackermann [5] treats only single input systems, with time invariant perturbations, whereas this report considers time varying uncertain parameters which vary within certain bounds. Lunze [6], using comparison systems concept and Owens and Chotai [7] by combining frequency domain and time domain treatments, present attractive schemes involving more general perturbations. Also Desoer et al [8] have established conditions for stability robustness of linear multivarible interconnected systems for sufficiently small perturbations.

From a different perspective, some researchers have presented analysis and design procedures for tolerable perturbtions for robust stability. Kantor and Andres [9] present an algorithm to determine tolerable perturbations in frequency domain using M matrix analysis. In time domain Horisberger and Belanger [10] present an algorithm to determine an output feedback control gain that yields the largest possible tolerable perturbation such that the closed loop system is stable, but no explicit bounds are given. Zheng [11] presents a procedure to find the stability regions as a function of parameters but considers only time invariant perturbations and no synthesis procedure is given. Eslami and Russell [12] also address the same problem but no explicit bounds are obtained. Chang and Peng [13], Patel and Toda [14], Patel, Toda and Sridhar [15] give explicit bounds on the linear, time varying perturbtions of an asymptotically stable linear time invariant system such that stability is guaranteed using Lyapunov theory. However the same measure is used to propose bounds for both 'structured' as well as 'unstructured' perturbations [16], making the bound for structured perturbation more conservative.

In what follows an analysis, carried out in the time domain, for both types of perturbation is presented. For structured perturbations, an improved upper bound is presented (over that of [14]), the expression for which is such that it garners the full 'structural' information about the nominal (and consequently, the perturbation) matrix into it. For unstructured perturbation case, some special cases of the nominally stable matrix are considered for which it is possible to give simplified upper bounds without the need to solve the Lyapunov equation. Examples presented include comparisons with the existing aproaches.

2.2.2 Perturbation Bounds for Linear State Space Models

Let us consider the following linear dynamic system,

$$\dot{x}(t) = A(t)\, x(t) \tag{2.1a}$$

$$= [A_o + E(t)]\, x(t) \tag{2.1b}$$

A_o is an nxn nominally asymptotically stable matrix and E is an nxn 'Error' matrix. In the case of 'structured' perturbation, the elements of E are such that

$$E_{ij}(t) < \varepsilon_{ij} \overset{\Delta}{=} \max_{t} |E_{ij}(t)| \quad \text{and } \varepsilon \overset{\Delta}{=} \max_{i,j} \varepsilon_{ij} \tag{2.2}$$

Thus ε is the magnitude of the maximum deviation expected in the entries of A. In the case of 'unstructured' perturbations only a 'norm' of the matrix E, say $\sigma_{max}(E)$ is assumed to be known.

2.2.1 Bounds for Structured Perturbation

For this situation, it is shown in [13], [14] that the system of (2.1) is stable if

$$\varepsilon < \frac{1}{n\, \sigma_{max}\, [P]} \equiv \mu_{\varepsilon P} \tag{2.3}$$

where P is the solution of the Lyapunov matrix equation

$$A_o^T P + P A_o + 2 I_n = 0 \quad (I_n \text{ is an nxn identity matrix}) \tag{2.4}$$

and it is assumed that $E_{ij} \neq 0$ for all i,j = 1,2...n.

At this stage, it is important to point out that Patel and Toda [14] do not formally exploit the distinction between the different types of perturbations as discussed in the Introduction and employ a single measure as the basis for an upper bound for all these types of perturbations. In this research, we clearly delineate this distinction and provide an improved upper bound for structured perturbations.

In what follows, we present the main mathematical result to get an upper bound for the perturbation matrix E of system (2.1), assuming structured perturbation.

Theorem 2.1: The system of (2.1) is stable if

$$\varepsilon < \frac{1}{\sigma_{max}\ [P_m U_n]_s} \equiv \mu_{\varepsilon Y} \tag{2.5}$$

where U_n is an nxn matrix whose entries are unity i.e., $U_{nij} = 1$ for all i,j = 1,...n and P satisfies the Lyapunov equation given by (2.4) and P_m is the matrix formed by the modulus of the entries of matrix P i.e., $P_{mij} = |P_{ij}|$.

Proof: Given in Appendix A.

However, the above expression for $\mu_{\varepsilon Y}$ is obtained without using the 'structural' information about the nominally stable matrix A_o. Obviously, one can get an even better bound if the structural information of the matrix A_o is taken into consideration. For example, as indicted by equation (2.2), if one knows the individual element bound ε_{ij} for all i,j, then this information can be incorporated into the calculation of the bound and in such cases, the bound $\mu_{\varepsilon Y}$ takes the following form.

<u>Theorem 2.2</u>: The system of (2.1) is stable if

$$\varepsilon < \frac{1}{\sigma_{max}\,(P_m U_e)_s} \equiv \mu_{Ye} \tag{2.6a}$$

where U_e is an nxn matrix whose entries are such that

$$U_{eij} = \frac{\varepsilon_{ij}}{\varepsilon} \tag{2.6b}$$

Thus $U_{eij} = 0$ if the perturbation in A_{oij} is known to be zero (i.e., $\varepsilon_{ij} = 0$). Similarly $U_{eij} = 1$ if one doesn't explicitly know ε_{ij} for some i,j thereby accommodating the worst case situation. Hence, it can be seen that

$$0 < U_{eij} \leq 1 \tag{2.7}$$

For example, if the nominally stable matrix is diagonal to start with and if it is known that there can be no perturbation possible in the off diagonal elements then U_e is taken to be I_n. Similarly if the nominally stable matrix A_o is such that

$$A_o = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \tag{2.8a}$$

and it is known that there cannot be any perturbations in a_{23}, a_{31} and a_{32} then

$$U_e = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.8b}$$

Similarly, if one also had additional information that $a_{ij\ min} < \bar{a}_{ij} < a_{ij\ max}$, then one can calculate $|\Delta a_{ij}| = \max\ (|a_{ij\ max} - \bar{a}_{ij})|,\ |(a_{ij\ min} - \bar{a}_{ij})|)$ for all i,j and determine $|\Delta a_{ij}|_{max}$. Accordingly, the matrix U_e elements will be

$$U_{eij} = \frac{|\Delta a_{ij}|}{|\Delta a_{ij}|_{max}} \tag{2.9}$$

Thus, the above expression (2.6) for μ_{Ye} garners the structural information of the nominally stable (and thereby the perturbation) matrix in a unified form, without the need for different expressions for different situations.

Remark 2.1: Since $\sigma_{max}\ (P_m U_e)_s < \sigma_{max}\ (P_m U_n)_s$ it can be seen that

$$\mu_{Ye} > \mu_{\varepsilon Y} \tag{2.10}$$

Remark 2.2 : The location of the perturbation (reflected by the location of nonzero entries in the U_e matrix) clearly influences the perturbation bound. Table 1 illustrates this point.

2.2.2. Bounds for Unstructured Perturbations

Now, we present similar 'stability robustness' measures assuming unstructured perturbations, namely perturbations where only the spectral norm bound for the error matrix (of a given model structure) is known [16].

Robustness Measure of Patel and Toda [14]

For this case, the condition for stability is given by

$$\sigma_{max}\ (E) = ||E||_s < \frac{1}{\sigma_{max}(P)} \equiv \mu_{PW} \tag{2.11}$$

where, as before, P satisfies equation (2.4). It is also shown that

$$\mu_{PW} = |\lambda\,(A_o)|_{min}$$

if A_o is normal thereby avoiding the solution of Lyapunov matrix equation.

Table 1. Elements of A_o in which perturbation is assumed.

$$\text{Let} \quad A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

	all a_{ij}	a_{11} only	a_{12} only	a_{21} only	a_{22} only	a_{11} & a_{12}	a_{11} & a_{22}	a_{11} & a_{21}	a_{12} & a_{21}	a_{12} & a_{22}	a_{21} & a_{22}	a_{11}, a_{12}, a_{21}	a_{11}, a_{12}, a_{22}	a_{11}, a_{21}, a_{22}	a_{12}, a_{21}, a_{22}
U_e	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
$\mu_{\gamma e}$	0.236	1.657	1.657	0.655	0.396	1.0	0.382	0.40	0.5	0.324	0.3027	0.397	0.311	0.273	0.256

Robustness Measure of Lee [17]

Assuming that the orthogonal matrix U in the polar decomposition of A,

$$A_o = UH_R \text{ or } A_o = H_L U$$

is a stable matrix, Lee gives a stability condition as follows:

$$\sigma_{max}(E) < -\sigma_{min}(A_o)\cos(\Theta_{min}) \equiv \mu_{LW} \quad (2.12)$$

where Θ_{min} is the smallest principal phase of A_o measured counter clockwise from the positive real axis. Also $\mu_{LW} = |\lambda(A_o)|_{min}$ if A_o is normal.

In the above two measures, the measure μ_{PW} doesn't exploit other forms of 'structural' information about A_o (except for the case of 'normal' matrix) whereas the measure μ_{LW} imposes a restriction on A (which in turn requires testing yet another condition). In what follows, exploiting other structural information about the nominally stable matrix A_o, we present a new stability robustness measure for unstructured perturbations.

Theorem 2.3: Let the nominal matrix A_o be stable and let $A_s = \dfrac{A_o + A_o^T}{2}$ be negative definite. Then the matrix $A_s + E_s$ is negative definite and hence (A_o+E) is stable, if

$$\sigma_{max}(E) < \sigma_{min}(A_s) \equiv \mu_{YW} \quad (2.13)$$

Proof: Given in Appendix B.

Note that there is no need to solve the Lyapunov equation to find μ_{YW}.

The following example compares the bounds μ_{PW}, μ_{LW} and μ_{YW} when the given nominal matrix is stable and A_s is negative definite (and A_o is not normal).

Example 2.1

$$A_o = \begin{bmatrix} -1 & -0.25 \\ +0.25 & -1.2 \end{bmatrix}$$

The eigenvalues of A_o are $-1.1 \pm j\ 0.229$

μ_{PW}	μ_{LW}	$\mu_{YW}[= \sigma_{min}(A_s)]$
1.0025	1.0025	1.0

In the above example even though μ_{YW} is less than μ_{PW}, there is considerable savings in the computation in its determination.

It may thus be seen that, depending on the structural information available about A_o, one can obtain different bounds. For completeness, we summarize in Table 2, the available bounds for unstructured perturbations and the related conditions under which they are obtained.

The main contribution of μ_{YW} pertains to Case (3) in the table. As noted, when the nominally stable matrix A_o is not normal but A_s is negative definite, one can get μ_{YW} quite easily wheres μ_{PW} and μ_{LW} require considerable computation. Of course, the bound μ_{YW} is more conservative than μ_{PW} because $\sigma_{min}(A_s) < \mu_{PW}$ as shown by Patel, Toda [14] and Yasuda, Hirai [18]. But example 2.1 above clearly demonstrates that, in some cases, μ_{YW} can offer appreciable savings in the computation without much sacrifice in the bound.

It may be helpful to mention here that one necessary condition on a stable matrix A_o such that A_s is negative definite is that $a_{ii} < 0$ for all $i = 1,2...n$. Incidentally, a sufficient condition is that A_o be normal. In fact, another coefficient condition can be proposed by applying theorems 2.1 and 2. 2 of Section 2.2. Writing

$$2A_s = A_o + A_o^T \tag{2.14}$$

and treating A^T as an 'error' matrix, we observe that A_s is stable (i.e., is negative definite) if

Table 2. Summary of Bounds for Unstructured Perturbations.

	Nominally stable Matrix A_o	Bounds on the error norm $\|\|E\|\|_s$ for stability	Remarks
1)	$A_o = A_s$ Symmetric (normal and negative definite	$\mu_{PW} = \mu_{LW} = \mu_{YW}$ $= \sigma_{min}(A_o)$	Simplest case but restrictive. No need for Lyapunov.
2)	A_s is negative definite and A_o is normal	$\mu_{PW} = \mu_{LW} = \|\lambda(A_o)\|_{min}$ $\mu_{YW} = \sigma_{min}(A_s)$	No need for Lyapunov in all the cases.
3)	A_s is negative definite, but A_o is not normal	$\mu_{PW} = 1/\sigma_{max}(P)$	Lyapunov solution needed.
		$\mu_{LW} = -\sigma_{min}(A_o)\ \cos\Theta_{min}$	Polar decomposition U to be stable.
		$\mu_{YW} = \sigma_{min}(A_s)$	Lyapunov solution not needed.
4)	General A_o (not normal) and A_s is not negative definite	$\mu_{PW} = 1/\sigma_{max}(P)$	Lyapunov solution needed.
		$\mu_{LW} = -\sigma_{min}(A_o)\ \cos\Theta_{min}$	Polar decomposition U to be stable.
			No μ_{YW} possible.

$$|A_o{}^T{}_{ij}|_{max} < \frac{1}{\sigma_{max}(P_m U_e)_s} \qquad (2.15)$$

where P satisfies $PA_o + A_o{}^T P + 2I_n = 0$ and $U_{e_{ij}} = \frac{|A_o{}^T{}_{ij}|}{|A_o{}^T{}_{ij}|_{max}}$ for all $i,j = 1,2...n$.

2.2.3 Application to an Aircraft Control Problem

We now consider the same application example as the one considered by Patel, Toda & Sridhar in [15]. For completeness sake, we briefly reproduce here the mathematical model of [15].

In [15], the system chosen is the flare control of the Augmentor Wing Jet STOL Research Aircraft (AWJSRA). The purpose of the flare control is to make a smooth transition from an initial steep flight path angle of -7.5^o on the glide slope at an altitude of approximately 65 ft. to a final smaller flight path angle (-1^o) more appropriate for touchdown.

The equations for the longitudinal dynamics of the AWJSRA at an airspeed of 110 ft/s and flight path angle of -1^o are given by

$$\dot{x} = A\,x + B\,u \qquad (2.16)$$

where $x^T = [\delta v \quad \delta\gamma \quad \delta\theta \quad \delta q \quad \delta h]^T$

$u^T = [\delta e \quad \delta n]^T$

δv = change in airspeed, ft/s

$\delta\gamma$ = change in flight path angle, deg

$\delta\theta$ = change in pitch angle, deg.

δq = change in pitch rate, deg/s

δh = deviation from nominal altitude, ft

δe = change in elevator deflection, deg

δn = change in nozzle angle, deg

$$A = \begin{bmatrix} -0.0547 & -0.298 & -0.2639 & -0.0031 & 0.0 \\ 0.16 & -0.4712 & 0.4661 & 0.0437 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.1751 & 0.1236 & -0.1236 & -1.3 & 0.0 \\ -0.0174 & 1.92 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{2.17a}$$

$$B = \begin{bmatrix} -0.00315 & -0.0943 \\ 0.0408 & 0.0224 \\ 0.0 & 0.0 \\ -1.1200 & -0.08 \\ 0.0 & 0.0 \end{bmatrix} \tag{2.17b}$$

The open loop poles of the system are at 0.0, $-0.0105 \pm j0.2737$, -0.6757 and -1.129. The control gain is obtained by minimizing [19] the performance index

$$J = \int_0^\infty (x^T Q x + u^T R u)dt \tag{2.18a}$$

with $R = \text{Diag}\ [16,\ 0.5]$ and $Q = qI_5$. (2.18b)

Applying the analysis of [15] and this paper, the bounds μ_Y (with U_n) and μ_P and their variation with q are summarized in Table 3 and Fig. 1.

Clearly it is seen that μ_Y is greater than μ_P for the values of q considered and the 'optimism' of μ_Y over μ_P increases as q is increased.

Table 3. Variation of μ_Y and μ_P with q

q	μ_P	μ_Y
0.1	0.0055	0.0061
0.25	0.0082	0.0093
0.5	0.0107	0.0125
1.0	0.0137	0.0164
5	0.0213	0.0272
10	0.0240	0.0322
50	0.0305	0.0420
10^2	0.0323	0.0451
10^4	0.0364	0.0530

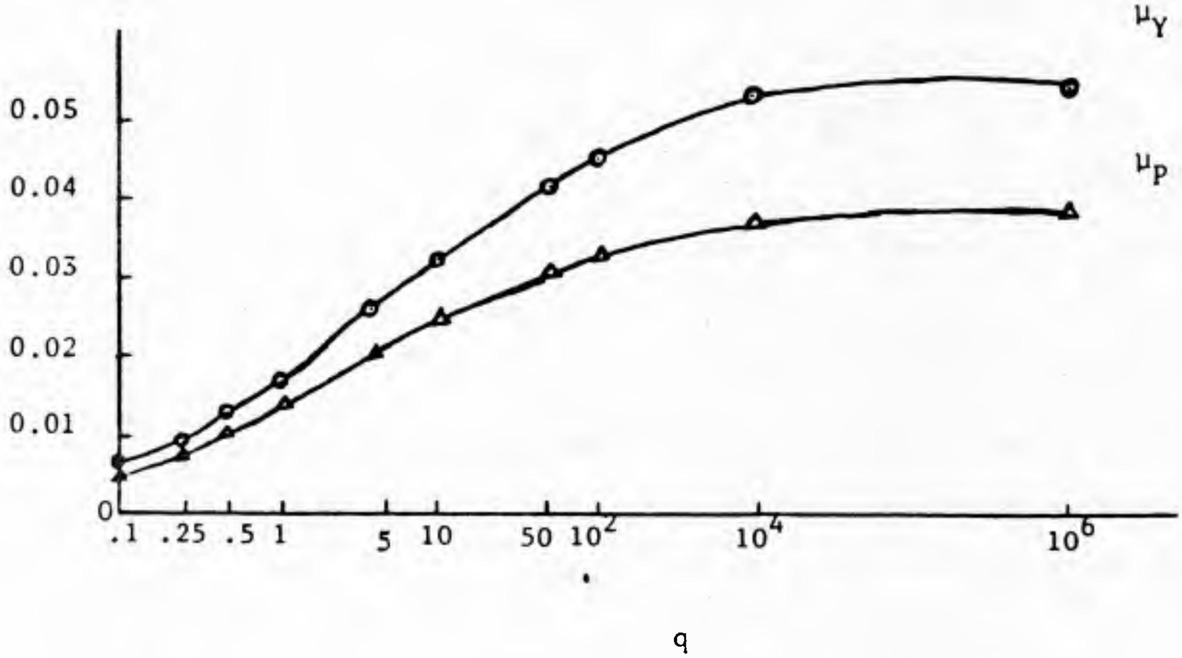


Fig. 1. Plot of μ vs. q

2.3 Reduction in Conservatism by State Transformation:

The proposed stability robustness measures presented in the previous section were basically derived using the Lyapunov stability theorem, which is known to yield conservative results. The 'improvement' obtained in the proposed bounds is the result of exploiting the 'structural' information about the perturbation. Clearly, another avenue available to further reduce the conservatism is to exploit the flexibility available in the construction of Lyapunov function used in the analysis. In this section, a method to further reduce the conservatism on the element bounds (for structural perturbation) is proposed by using state transformation. This reduction in conservatism is obtained by exploiting the variance of the 'Lyapunov criterion conservatism' with respect to the basis of the vector space in which the function is constructed. The proposed transformation technique seems to almost always increase the region of guaranteed stability and thus is found to be useful in many engineering applications.

2.3.1 State Transformation and its Implications on Bounds

It may be easily shown that the linear system (2.1) is stable (or asymptotically stable) if and only if the system

$$\dot{\hat{x}}(t) = \hat{A}(t)\,\hat{x}(t) \tag{2.19a}$$

where

$$\hat{x}(t) = Q^{-1}\,x(t),\quad \hat{A}(t) = Q^{-1}\,A(t)Q \tag{2.19b}$$

and Q is a nonsingular time invariant nxn matrix, is stable (or asymptotically stable).

Even though the proof of this result is quite straightforward, it is to be emphasized that it is not based on the standard eigenvalue argument as we are dealing with time varying case.

The implication of this theorem is, of course, important in the proposed analysis here. It means that, to investigate the stability of a linear system of (2.1), one can transform it, by a linear map, to a different coordinate frame and derive the stability robustness condition in the new (transformed) coordinates. Since the available stability robustness conditions are just sufficient conditions (in contrast to necessary and sufficient conditions), it is likely that one can get a less conservative bound in the transformed coordinates.

In sequel, we consider a diagonal transformation matrix Q. (The use of a general non-diagonal transformation matrix is under study.) Let

$$Q = \mathrm{diag}[q_1, q_2, \ldots q_n] \quad q_i \neq 0, \; i = 1,2,\ldots,n \tag{2.20}$$

Then $\hat{A}(t) = \hat{A}_o + \hat{E}(t) =$

$$\begin{bmatrix} a_{11}+e_{11}(t) & q_2/q_1(a_{12}+e_{12}(t)) \ldots\ldots & q_n/q_1(a_{1n}+e_{1n}(t)) \\ q_1/q_2(a_{21}+e_{21}(t)) & a_{22}+e_{22}(t) \ldots\ldots\ldots\ldots & q_n/q_2(a_{2n}+e_{2n}(t)) \\ \vdots & \vdots \quad \cdots\cdots & \vdots \\ q_1/q_n(a_{n1}+e_{n1}(t)) & q_2/q_n(a_{n2}+e_{n2}(t)) \ldots\ldots\ldots\ldots & (a_{nn}+e_{nn}(t)) \end{bmatrix} \tag{2.21a}$$

where $\hat{A}_o = Q^{-1}A_oQ$ and $\hat{E}(t) = Q^{-1}E(t)Q$. (2.21b)

Correspondingly, we get

$$\hat{\varepsilon}_{ij} = \left|\frac{q_j}{q_i}\right| \varepsilon_{ij} \;, \text{ and } \hat{\varepsilon} = \underset{i,j}{\mathrm{Max}}\; \hat{\varepsilon}_{ij} \tag{2.22}$$

and as discussed before

$$U_{eij} = \hat{\varepsilon}_{ij}/\hat{\varepsilon} \tag{2.23}$$

It may be seen that (2.21) is similar in form to the 'weighted norm' matrix which has been used successfully in frequency domain to reduce the conservatism of the stability robustness condition [20, 21, 22].

In general, stability robustness conditions of the type given in (2.6) may be used in two different situations: (a) given the perturbation element bounds, ε_{ij} (of the l.h.s. of equation (2.6), the condition (2.6) can be used to check the stability of the perturbed system (2.1) as well as check the correponding conservatism of the condition, or (b) simply treat the condition (2.6) as specifying a bound on the elemental perturbation (i.e., ε_{ij} of l.h.s. of (2.6) are not explicitly known but U_e of r.h.s. of (2.6) is known; recall that in the absence of any explicit and relative information on ε_{ij}, one can take $U_{eij} = 1$ thereby accounting for the worst case situation). This type of delineation is useful in arriving at a transformation appropriate for the purpose at hand. We illustrte these situations by way of simple examples.

(a) L.H.S. is known (checking stability and conservatism):

For this case, in the transformed coordinates, the stability condition (2.6) bcomes

$$\hat{\varepsilon}_{ij} < \hat{\mu}\, \hat{U}_{eij} \tag{2.24a}$$

or

$$\hat{\varepsilon} < \hat{\mu} \tag{2.24b}$$

where

$$\hat{\mu} = 1/\sigma_{max}\, [\hat{P}_m \hat{U}_e]_s \tag{2.25a}$$

and $\hat{P}$ satisfies $\hat{P}\hat{A}_o + \hat{A}_o\hat{P} + 2I_n = 0$ (2.25b)

The conservatism of the condition with respect to the transformation can clearly be compared by using the index, defined by

$$\beta \overset{\Delta}{=} \frac{\mu-\varepsilon}{\mu} \qquad \hat{\beta} \overset{\Delta}{=} \frac{\hat{\mu}-\hat{\varepsilon}}{\hat{\mu}} \tag{2.26}$$

where it can be seen that β and $\hat{\beta} > 0$ when the stability condition gets

satisfied and that

$$\hat{\beta} > \beta \qquad (2.27)$$

indicates the transformation to be effective in reducing the conservatism of the condition. Let us illustrate this by an example.

Example 2.2: Let A_o of (2.1) be given by

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

Suppose that only element a_{11} gets perturbed and that

$$\varepsilon_{11} = |\varepsilon_{11}|_{max} = 2$$

Then clearly

$$\varepsilon = 2 \text{ and } U_e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The r.h.s. of equation (2.6) gives $\mu = 1.657$. Thus

$$\beta = \frac{1.657 - 2}{1.657} = -0.207$$

is negative indicating that the stability condition is not satisfied. However, employing the transformation matrix

$$Q = \text{diag } [1, \quad 1000]$$

the transformed quantities are

$$\hat{\varepsilon} = 2 \text{ and } \hat{U}_e = U_e \text{ and } \hat{\mu} = 3.$$

Thus

$$\hat{\beta} = \frac{3 - 2}{3} = 0.333$$

indicating that the stability condition is satisfied and hence the system (even in the original coordinates) is stable. Thus the transformation reduced the conservatism of the stability condition.

(b) L.H.S. is not known (specifying the perturbation bound):

As before in case (a), after the transformation, the stability condition is

$$\hat{\varepsilon} < \hat{\mu} \tag{2.28}$$

However, as a bound, $\hat{\mu}$ is given on the perturbation $\hat{\varepsilon}$ and not on ε, the perturbation in the original coordinates. Evidently, in order to examine the usefulness of the transformation in getting an improved bound it is necessary to obtain the bound on ε after the transformation. Let μ^* denote the bound on ε after the transformation. Then we have the following result:

Theorem 2.4 : Given $Q = \text{diag}\,[q_1, q_2, \ldots, q_n]$, and eq. (2.22) the system of (2.1) is stable if

$$\varepsilon < \mu^* \tag{2.29a}$$

$$\mu^* = \hat{\mu}\,\frac{1}{U_{ers}}\left|\frac{q_r}{q_s}\right| \tag{2.29b}$$

where $\hat{\mu}$ is given by (2.25) and r,s are such that rs is the specific entry in $\hat{U}_e$ corresponding to $\hat{U}_{eij\,\max}$ and $\hat{U}_e = Q_m^{-1} U_e Q_m$.

Remark 2.3: Clearly $\mu^* > \mu$ indicates the reduction in conservatism of the condition.

Remark 2.4: It can also be noted that

$$\mu^* > \mu \tag{2.30}$$

if and only if (proof is available)

$$\hat{\beta} > \beta \tag{2.31}$$

where $\hat{\beta}$ and β are given by (2.26).

Example 2.3: Let us again consider

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

and let

$$U_e = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} . \quad \text{Then we have } \mu = 0.3972.$$

By using the transformation $Q = \text{diag}\ [1,\ 1.8]$ it can be shown that

$$\mu^* = 0.473$$

indicating around 20% improvement in the bound.

2.3.2 Application to VTOL Aircraft Control

In this section, we extend the proposed method of reducing the conservatism by state transformation to the VTOL aircraft control example discussed in [23]. This particular aircraft control problem has been used in the illustrtion of several control design methodologies. For example [23] suggests an adaptive control algorithm for the given range of perturbation while [29] recommends a nonlinear feedback control to achieve boundedness which, however, still cannot guarantee robustness for the entire range of perturbation considered. In the present discussion, we propose a simple constant linear rate feedback to guarantee asymptotic stability using the proposed perturbation bound analysis with state transformation.

The linearized model of the VTOL aircraft in the vertical plane is described by

$$\dot{x}(t) = (A_o + \Delta A(t))\ x(t) + (B + \Delta B(t))u(t) \tag{2.32}$$

The components of the state vector $x \to R^4$ and the control vector $u \to R^2$ are given by

$x_1 \rightarrow$ horizontal velocity (knots)

$x_2 \rightarrow$ vertical velocity (knots)

$x_3 \rightarrow$ pitch rate (degrees/sec)

x_4 $\rightarrow$ pitch angle (degrees)

u_1 $\rightarrow$ 'collective' pitch control

u_2 $\rightarrow$ 'longitudinal cyclic' pitch control

In [23], it is shown that significant changes take place only in the elements a_{32}, a_{34} and b_{21}. The ranges of values taken by these elements are given in [23] as

$$
\begin{aligned}
0.0663 &\leq a_{32}(\sim 0.3681) \leq 0.5044 \\
0.122 &\leq a_{34}(\sim 1.422) \leq 2.528 \\
0.977 &\leq b_{21}(\sim 3.544) \leq 5.1114
\end{aligned}
\tag{2.34}
$$

Thus the perturbation bounds are asymmetric with respect to the nominal value. In order to take full advantage of the 'Perturbation Bound Analysis' presented in Section II, we will 'bias' the nominal value of a_{32}, a_{34} and b_{21} such that we obtain symmetric bounds. Accordingly, the nominal values of a_{32}, a_{34} and b_{21} are $\bar{a}_{32} = 0.2855$, $\bar{a}_{34} = 1.3229$, $\bar{b}_{21} = 3.04475$. The full matrices A_o and B_o are given by

$$
A_o = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.35a}
$$

$$
B_o^T = \begin{bmatrix} 0.4422 & 3.04475 & -5.52 & 0 \\ 0.1761 & -7.5922 & 4.49 & 0 \end{bmatrix} \tag{2.35b}
$$

so that

$$|\Delta A_{32}|_{max} = 0.2197$$

$$|\Delta A_{34}|_{max} = 1.2031$$

$$|\Delta A_{ij}| = 0 \text{ for all other } i \text{ and } j \tag{2.36}$$

$$|\Delta B_{21}|_{max} = 2.06725$$

$$|\Delta B_{ij}|_{max} = 0 \text{ for all other } i \text{ and } j$$

A Robust constant gain linear state feedback control law is obtained as follows:

Step 1: Assuming a quadratic performance index

$$J = \int_{0}^{\infty} (x^T x + \rho u^T u)dt \; , \tag{2.37}$$

where ρ is the control weighting, the optimal feedback gain G_1 is obtained as

$$G_1 = -\frac{1}{\rho} B_o{}^T K \tag{2.38a}$$

where

$$KA_o + A_o{}^T K - \frac{K\, B_o B_o{}^T K}{\rho} + I_4 = 0 \tag{2.38b}$$

and the closed loop nominal matrix

$$\bar{A} = A_o + B_o G_1 \tag{2.39}$$

is found to be asymptotically stable as the conditions of complete controllability and observability are satisfied.

The perturbed closed loop system is then given by

$$\dot{x}(t) = [(A_o + B_o G_1) + (\Delta A + \Delta B\, G_1)]\, x\,(t) \tag{2.40a}$$

$$= [(A_o + B_o G_1) + E]\, x\,(t) \tag{2.40b}$$

where $E_m = \Delta A_m + \Delta B_m G_{1m}$. Let $\varepsilon_{ij} \overset{\Delta}{=} E_{mij}$.

Step 2: Clearly in this case the perturbation matrix is fully known. The variable ρ is varied such that the stability robustness index

$$\beta \triangleq \frac{\mu(G_1) - \varepsilon(G_1)}{\mu(G_1)} \tag{2.41}$$

is made as large as possible where μ is given by

$$\mu = 1/\sigma_{max}[P_m U_e]_s \tag{2.42a}$$

and P_m satisfies

$$P(A_o+B_oG_1) + (A_o+B_oG_1)^T P + 2I_n = 0 \tag{2.42b}$$

(Note that both μ and ε are functions of gain G_1).

The gain G_1 that makes β of (2.41) maximum for the given perturbations (2.36) is obtained as

$$G_1 = \begin{bmatrix} -0.467 & 0.01388 & 0.539 & 0.806 \\ 0.043 & 0.3828 & -0.1899 & -0.5947 \end{bmatrix} \tag{2.43}$$

and the corresponding value of β is given by

$$\beta = -0.21 \tag{2.44}$$

which shows that the stability robustness condition is not satisfied with the gain G_1 of (2.43).

Step 3: Evidently, one needs to look for a gain such that β is positive. In other words, one needs to increase the bound μ without much increase in ε in equation (2.6). Toward this direction, for this particular example, we propose to use a gain G_2 such that

$$A_o + B_oG_1 + B_oG_2 = A_o + B_oG \quad (G \triangleq G_1 + G_2) \tag{2.45}$$

is asymptotically stable and

$$\tilde{E}_m = \Delta A_m + \Delta B_m G_{1m} + \Delta B_m G_{2m} = E_m \quad (\text{i.e.,}\quad G_m G_{2m} = 0) \tag{2.46}$$

One form for G_2 that satisfies (   ) since only B_{21} is nonzero) is

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ g_{21} & g_{22} & g_{23} & g_{24} \end{bmatrix}$$

The motivation behind the selection of a specific G_2 matrix as above is to make the norm of the matrix A_o+B_oG bigger with the hope of decreasing the norm of the Lyapunov matrix P of (2.42) which in turn increases the bound μ without increasing ε of (2.6). With this in mind, we obtain $g_{21} = g_{23} = 0$ and $g_{22} > 0$ and $g_{24} < 0$. For simplicity, we choose $|g_{22}| = |g_{24}|$.

To guarantee the asymptotic stability of A_o+B_oG of (2.45) we could think of the matrix B_oG_2 as a perturbation on the nominal matrix $A_o+B_oG_1$ and apply the perturbation bound condition of (2.6). With this done, we get

$$7.6g_{22} < 1.0342$$

which makes $g_{22} = 0.1362$ and $g_{24} = -0.1362$.

Thus, we finally get

$$G = G_1+G_2 = \begin{bmatrix} -0.467 & 0.01388 & 0.539 & 0.806 \\ 0.043 & 0.519 & -0.1899 & -0.731 \end{bmatrix} \tag{2.47a}$$

and

$$A_o+B_oG = \begin{bmatrix} -0.2356 & 0.1246 & 0.22377 & -0.2277 \\ -1.7021 & -4.908 & 3.0859 & 3.98 \\ 2.8732 & 2.539 & -4.5359 & -6.408 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.47b}$$

Now the computation of β of (2.41) with this new closed loop system gives

$$\beta_{new} = -\,0.1722 \tag{2.48}$$

which is still negative but is an improvement over the β obtained before. However, one cannot conclude the stability of the new perturbed closed loop system using this β_{new}.

Step 4: We now apply state transformation for the above nominal system using a transformation matrix

$$Q = \text{diag } [0.7, 1.6, 1.02, 0.8] \tag{2.49}$$

The computation of $\hat{\beta}_{new}$ (after the transformation) gives

$$\hat{\beta}_{new} = 0.029 > 0$$

Thus, the state transformation given by the matrix Q of (2.49) has reduced the conservatism of the stability condition and the gain matrix given by (2.47) guarantees robust asymptotic stability of the system in the entire range of parameter perturbations given by (2.34) (or (2.36)).

2.4 Reduced Conservatism in Ultimate Boundedness Control

In this section, we apply the concept of reducing conservatism by state transformation to the problem of obtaining stabilizing controllers for uncertain dynamical systems using the theory of ultimate boundedness notably papers by Leitmann, Barmish and their colleagues [24,25,26,27,28].

The basic philosophy of ultimate boundedness control is that it guarantees the state of the system to enter and remain within a prescribed neighborhood of the origin (zero set point) or even a nonzero set point. In refs. [24-25] controllers are proposed that guarantee ultimate boundedness for systems satisfying the so-called 'matching conditions' namely, certain preconditions on the location and magnitude of the uncertainty within the system matrices. Under the matching condition assumption uncertainties with an arbitrarily large prescribed bounds can be tolerated. Barmish [26] extended this theory to more general situations by introducing the concept of 'Measure of Mismatch' (and 'Mismatch Threshold').

In this framework, a certain 'decomposition' is performed on the system dynamics which yields a 'matched portion'. It is shown that effective control is possible as long as the 'measure of mismatch' does not exceed some critical 'mismatch threshold'. Even though this 'mismatched uncertainty' increases the applicability of the theory of ultimate boundedness to more general forms of uncertainty, it turns out that the resulting bounds are very small and thus conservative. The reason for this conservatism is twofold. Firstly, the decomposition of the system into matched and mismatched parts is nonunique; the 'better' the decomposition, 'larger' the size of the uncertainty that can be tolerated. Secondly, the 'mismatch threshold' is calculated based on 'Lyapunov stability theory' which is always known to be conservative in predicting stability, as discussed in the previous section.

The main focus of this section is to address these aspects of i) consevatism in the bounds obtained in time domain using Lyapunov theory and ii) the nonuniqueness of the decomposition. Specifically, a diagonal similarity transformation on the system is proposed which serves to achieve either a less conservative bound on the mismatched perturbation or a 'better' decomposition of the matched and mismatched parts or both. The proposed extension can be applied to both zero set point as well as nonzero set point control [29].

2.4.1 System Description and Transformation on Boundedness Set

As in refs. [26,29], consider the uncertain dynamic system described by the state equation

$$\dot{x}(t) = [A+\Delta A(r(t))]x(t) + [B+\Delta B(r(t))]u(t)+Cv(t) \tag{2.50}$$

where $x(t)\to R^n$ is the state, $u(t)\to R^m$ is the control and $v(t)\varepsilon R^s$ is the

disturbance input. A,B,C are nominal system matrices of appropriate dimensions and ΔA and ΔB are the uncertainty matrices depending continuously on the uncertain parameter vector $r(t) \rightarrow R^p$. Also

$$r(\cdot) \rightarrow \Omega \rightarrow R^p$$

$$v(\cdot) \rightarrow V \rightarrow R^s$$

are Lebesque measurable, where Ω and V are prescribed compact subsets of appropriate spaces. It is also assumed that the pair (A,B) is completely controllable. Let us also define

$$\begin{aligned} z(t) &= x(t) - x^*, \; x(t_o) = x_o \\ z(t_o) &\overset{\Delta}{=} z_o = x_o - x^* \end{aligned} \tag{2.51}$$

where x^* is the set point of the state. We denote the solution of (2.50) by $x(t,x_o,t_o)$. Let $||(\cdot)||$ denote the Euclidean norm for a vector and spectral norm $[=\lambda_{max}(\cdot)^T(\cdot)]$ for a matrix $(\cdot)$.

We now restate the definitions used in [26] for continuity and better exposition.

Definition 1: Given a solution $x(t,x_o,t_o)$, $t \rightarrow [t_o,t_1]$ of (2.50) and a set point $x^* \rightarrow R^n$, we say that the solution is uniformly bounded if there is a positive constant $d(z_o) < \infty$, possibly dependent on z_o, but not on t_o, such that

$$||x(t,x_o,t_o)-x^*|| \leqslant d(z_o) \text{ for all } t \rightarrow [t_o,t_1].$$

Definition 2: Given a solution $x(t,x_o,t_o)$, $t \rightarrow (t,\infty)$ of (2.50) and a set point $x^* \rightarrow R^n$, we say that the solution is uniformly ultimately bounded with respect to a set $\hat{S}$ if there is a nonnegative constant $T(z_o,\hat{S}) < \infty$, possibly dependent on z_o and $\hat{S}$ but not on t_o, such that $x(t,x_o,t_o) \rightarrow \hat{S}$ for all $t \geqslant t_o+T(z_o,\hat{S})$ and that $x^* \rightarrow \hat{S}$.

We now present the following theorems which are crucial in justifying the use of a transformation for later analysis.

Theorem 2.5: $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is uniformly bounded to the set point $x^* = [x^*_1, x^*_2, \ldots, x^*_n]^T$ if and only if

$$\bar{x}(t) = [\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t)]^T$$
$$= [k_1 x_1(t), k_2 x_2(t), \ldots, k_n x_n(t)]^T \tag{2.52}$$

k_i = constant $\neq 0 \ \forall \ i=1,\ldots n.$

is uniformly bounded to the set point

$$\bar{x}^* = [k_1 x^*_1, k_2 x^*_2, \ldots, k_n x^*_n]^T$$

Theorem 2.6: $x(t)$ is uniformly ultimately bounded with respect to a set

$$\hat{S} \stackrel{\Delta}{=} (x \to R^n: \ (x(t)-x^*)^T \ P(x(t)-x^*) < K) \tag{2.53}$$

where P is a symmetric positive definite matrix, if and only if $\bar{x}(t)$ of (2.52) is uniformly ultimately bounded with respect to a set

$$\hat{S} \stackrel{\Delta}{=} (\bar{x} \to R^n: \ (\bar{x}(t)-\bar{x}^*)^T \ \bar{P}(\bar{x}(t)-\bar{x}^*) < \bar{K}) \tag{2.54}$$

These two theorems aid us now to establish the transformation

$$x = Q\bar{x} \tag{2.55}$$

where $Q = \text{diag}\ [\ldots q_i \ldots]$, $q_i = 1/k_i \ \forall \ i = 1\ldots n$ to be a similarity transformation. With this transformation the system of (2.50) is transformed to

$$\dot{\bar{x}}(t) = (\bar{A}+\Delta\bar{A}(r(t))\bar{x}(t) + (\bar{B}+\Delta\bar{B}(r(t))u(t)+\bar{C}v(t) \tag{2.56}$$

where

$$\bar{A} = Q^{-1}AQ, \qquad \Delta\bar{A} = Q^{-1}\Delta AQ$$
$$\bar{B} = Q^{-1}B, \qquad \Delta\bar{B} = Q^{-1}\ \Delta B \text{ and } \bar{C} = Q^{-1}C \tag{2.57}$$

We now recall from theorems 2.5 and 2.6 that if the solution of (2.50) is uniformly bounded and uniformly ultimately bounded with respect to a specific set, then so is the solution of (2.56). We also note from [26] that the main condition for a solution of (2.50) to be bounded is the so called mismatch constraint condition

$$\tilde{M} < M^* \tag{2.58}$$

where $\tilde{M}$ is the 'measure of mismatch' and M^* is the 'mismatch threshold'. The motivation behind employing the transformation (2.55) is to basically transform the above mismatch condition with the hope that it would yield an improvement over the original condition (2.58) in the following sense:

Define β, the 'Mismatch Conservatism Index' (MCI) in the original system as

$$\beta \stackrel{\Delta}{=} (M^* - \tilde{M})/M^* \tag{2.59}$$

and $\bar{\beta}$, the MCI for the transformed system

$$\bar{\beta} \stackrel{\Delta}{=} (\bar{M}^* - \tilde{\bar{M}})/\bar{M}^* \tag{2.60}$$

Then if $\bar{\beta} > \beta$, we have a less conservative mismatch constraint condition.

2.4.2 Control Strategy and Mismatch Condition Under Transformation

The nonlinear control law obtained for the original system [26] (for simplicity, taking the set point x^* to be zero) is given by

$$u(t) = Gx + p(x) \tag{2.61}$$

where the constant gain matrix G is the linear feedback gain that makes

$$A_c \stackrel{\Delta}{=} A + BG \tag{2.62}$$

stable. Following the lines of [26], the decomposition of the uncertainty matrices into matched and mismatched portions is given by

$$\begin{aligned} \Delta A(r(t)) &= \Delta A_m(r(t)) + \Delta\tilde{A}(r(t)) = BD(r(t)) + \Delta\tilde{A}(r(t)) \\ \Delta B(r(t)) &= \Delta B_m(r(t)) + \Delta\tilde{B}(r(t)) = BE(r(t)) + \Delta\tilde{B}(r(t)) \\ C &= C_m + \tilde{C} = BF + \tilde{C} \end{aligned} \quad (2.63)$$

Provided that

$$\tilde{E} \overset{\Delta}{=} [1 - \underset{r \to R}{\mathrm{Max}} \, ||E(r(t))||] > 0$$

we define

$$\lambda_2 = \tilde{E}^{-1} \, [\underset{r \to R}{\mathrm{Max}} ||D(r(t))|| + \underset{r \to R}{\mathrm{Max}} ||E(r(t))G||]$$

and furthermore

$$\tilde{M} = \underset{r \to R}{\mathrm{Max}} ||\Delta\tilde{A}(r(t)|| + \underset{r \to R}{\mathrm{Max}} ||\Delta\tilde{B}(r(t))G|| + \lambda_2 \underset{r \to R}{\mathrm{Max}} ||\Delta\tilde{B}(r(t))||$$

Then, the mismatch constraint condition for the original system (2.50) is

$$\tilde{M} < M^*$$

where $M^* = 1/(2\lambda_{max}(P))$ (2.64a)

and P is the symmetric positive definite matrix obtained as the solution of the Lyapunov equation

$$PA_c + A_c^T P + I_n = 0 \quad (2.64b)$$

It may be noted that M^* as given by eq. (2.64) is exactly the same quantity obtained by Patel, Toda in [14] as the bound for the nonlinear (as well as linear unstructured) perturbation (on an asymptotically stable linear system) that guarantees the stability of the perturbed system.

Now applying the suggested transformation (2.55) on (2.61) and it gives

$$u = \bar{G}\,\bar{x}(t) + \bar{p}(\bar{x}) \quad (2.65)$$

and

$$\dot{\bar{x}}(t) = (\bar{A}+\Delta\bar{A}(r(t))\bar{x}(t) + (\bar{B}+\Delta\bar{B}(r(t))(\bar{G}\ \bar{x}(t)+\bar{p}(\bar{x})) + \bar{C}v(t) \tag{2.66a}$$

where $\bar{G} = GQ$ (2.66b)

and $\bar{p}(\bar{x}) = p(Q\ \bar{x})$ (2.66c)

Let the decomposition be given by

$$\Delta\bar{A}(r(t)) = \Delta\bar{A}_m(r(t)) + \Delta\tilde{\bar{A}}(r(t)) = \bar{B}\ \bar{D}\ (r(t)) + \Delta\tilde{\bar{A}}(t(t))$$

$$\Delta\bar{B}(r(t)) = \Delta\bar{B}_m(r(t)) + \Delta\tilde{\bar{B}}(r(t)) = \bar{B}\ \bar{E}(r(t)) + \Delta\tilde{\bar{B}}(r(t)) \tag{2.67}$$

$$\bar{C} = \bar{C}_m + \tilde{\bar{C}} = \bar{B}\ \bar{F} + \tilde{\bar{C}}$$

Provided that

$$\tilde{\bar{E}} = (1 - \max_{r\to R} ||\bar{E}(r(t))||) > 0$$

we define

$$\lambda_2 = \tilde{\bar{E}}^{-1}\ (\operatorname*{Max}_{r\to R}||\bar{D}(r(t)|| + \operatorname*{Max}_{r\to R}||\bar{E}(r(t))\bar{G}||)$$

and furthermore

$$\tilde{\bar{M}} = \operatorname*{Max}_{r\to R}||\Delta\tilde{\bar{A}}(r(t)|| + \operatorname*{Max}_{r\to R}||\Delta\tilde{\bar{B}}(r(t))\bar{G}|| + \bar{\lambda}_2 \operatorname*{Max}_{r\to R} ||\Delta\tilde{\bar{B}}(r(t))||$$

and the mismatch constraint condition for this transformed system is

$$\tilde{\bar{M}} < \bar{M}^* \tag{2.68}$$

where

$$\bar{M}^* = 1/(2\lambda_{max}(\bar{P})) \tag{2.69a}$$

where $\bar{P}$ satisfies

$$\bar{P}\ \bar{A}_c + \bar{A}_c^T\bar{P} + I_n = 0 \tag{2.69b}$$

and

$$\bar{A}_c = \bar{A} + \bar{B}\,\bar{G} \tag{2.69c}$$

Note that $\bar{A}_c = Q^{-1}A_cQ$ is stable whenever A_c is stable.

Now, it may be observed from the above discussion that there are two avenues available for reducing the conservatism of the mismatch condition by transformation. One is to get a 'better' decomposition through eq. (2.67) than the original decomposition of (2.63) in the sense that bigger portions of the uncertainties may be lumped into the matched part in the transformed equations. The other avenue available is to get an improved bound M* (in the transformed system) since conservatism of the Lyapunov condition is <u>not</u> invariant under transformation. However, it is interesting to note that by the specific transformation suggested in (2.55), we may be able to achieve both (better decomposition as well as better bound) whereas in some situations we may achieve either one of them. In some other situations whatever gained in one avenue may be lost in the other. Thus, in order to assess the final effectiveness of any proposed transformation, one has to compare β and $\bar{\beta}$ (the Mismatch Conservatism Index) defined earlier.

Once the mismatch constraint condition is analyzed in the transformed system for possible reduction in conservatism, it is desirable to express the control law and the ultimate boundedness measures like the radius of the largest closed ball wholly contained in the set (denoted by η in [26] and η_{max} in [29]) in the original coordinates. This aspect is discussed in the next section.

<u>2.4.3 Control Law and Ultimate Boundedness Measures w.r.t. The Original System</u>

It may be observed that the mismatch constraint condition is related to the decomposition of the uncertainties and thus is related to the nonlinear control

law as well. Since the decomposition employed in the transformed system to get a 'better' mismatch condition is generally different from the one obtained by transformation on the original decomposition, i.e., $\overline{D}$, $\overline{E}$ and $\overline{F}$ of expression (2.67) are not necessrily similar to D, E and F of expression (2.63) before the transformation Q (or Q^{-1}), the parameters of the control law in the original coordinates obtained by inverse transformation could be different from those of the control law in the original system before transformation.

Control Law

Suppose the control law in the transformed system is found as

$$u = \overline{G}^* \, \overline{x} + \overline{p}(\overline{x}) \tag{2.70}$$

then, by inverse transformation, the control law in the original coordinates, is

$$u = G_n x + p_n(x) \tag{2.71a}$$

where

$$G_n = \overline{G}^* \, Q^{-1} \;, \; p_n(x) = \overline{p}(Q^{-1} \, x) \tag{2.71b}$$

and the matrix $A+BG = Q(\overline{A} + \overline{B} \, \overline{G}^*)Q^{-1}$ is stable because $A + BG^*$ is stable by design. The nonlinear function $p_n(x)$ can be written as [26]

$$p_n(x) = \begin{cases} - \dfrac{\overline{B}^T P Q^{-1} x}{||\overline{B}^T PQ^{-1} x||} \, \overline{\rho}(Q^{-1} x), \text{ if } ||\overline{B}^T P Q^{-1} x|| > \overline{\varepsilon} & (2.72a) \\[2ex] - \dfrac{\overline{B}^T P Q^{-1} x}{\overline{\varepsilon}} \, \rho(Q^{-1} \, x), \text{ if } ||\overline{B}^T P Q^{-1} x|| < \overline{\varepsilon} & (2.72b) \end{cases}$$

where $\rho(Q^{-1} \, x) = \lambda_1 + \lambda_2 ||Q^{-1} x||$

and

$$\lambda_1 = [1 - \max_{r \to R} ||\bar{E}(r(t)||]^{-1} \max_{v \to V} ||Fv||$$

$$\lambda_2 = [1 - \max_{r \to R} ||\bar{E}(r(t)||]^{-1} (\max_{r \to R} ||\bar{D}(r(t)|| + \max_{r \to R} ||\bar{E}(r(t))\bar{G}||)$$

<u>Ultimate Boundedness Measures (U.B.M.)</u>

One rough measure of the ultimate boundedness set, used for comparisons, is the radius of the largest closed ball wholly contained within the ultimate boundedness set S. This is denoted by η in [29] and η_{max} in Barmish [26]. Let us denote it by η. Recall that if the solution of (2.50) is uniformly ultimately bounded with respect to a set S(K)

$$\hat{S}(K) = \{x \to R^n: (x(t)-x^*)^T P(x(t)-x^*) \leq K\} \tag{2.73}$$

then η and K are related by [26]

$$K = \lambda_{max}(P)\eta^2 \tag{2.74}$$

From the theorems 2.5 and 2.6 we can observe that if the solution of the transformed system (2.56), $\bar{x}(t)$, is uniformly ultimately bounded with respect to a set

$$\tilde{\bar{S}}(K) = \{\bar{x} \to R^n: (\bar{x}(t)-\bar{x}^*)^T \bar{P}(\bar{x}(t)-\bar{x}^*) \leq \bar{K}\} \tag{2.75}$$

where $\bar{K} = \lambda_{max}(\bar{P})\bar{\eta}^2$ (and $\bar{\eta}$ is the U.B.M. in the transformed system) then the solution of the original system $x(t) = Q\bar{x}(t)$ is uniformly ultimately bounded with respect to a set

$$\hat{S}(K_n) = \{x \to R^n: (x(t)-x^*)^T P(x(t)-x^*) \leq K_n \tag{2.76a}$$

where

$$K_n = \lambda_{max}(\bar{P})/\lambda_{min}(\bar{P})(\max_i |q_i|)^2 \bar{K} \tag{2.76b}$$

Note that in (2.76) the weighting matrix in the boundedness set with

original coordinates is $\tilde{P}$, instead of P, the weighting matrix in the set of the original system before transformation. In fact, it is one viewpoint that accounts for why we can expect to get a better mismatch constraint condition.

From eqs. (2.74), (2.76), we observe that the radius of the largest closed ball wholly contained within the ultimate boundedness set $\hat{S}(K_n)$, denoted by η_n, can now be related to K_n as

$$K_n = \lambda_{max}(P)\eta_n{}^2 \tag{2.77}$$

Thus, from (2.75), (2.76b), we get

$$\eta_n = [\lambda_{max}(\overline{P})/\lambda_{min}(\overline{P})]^{1/2}\ (\max_i |q_i|)\overline{\eta} \tag{2.78}$$

Hence, once $\overline{\eta}$ (in the transformed system) is known, one can get η_n. Of course from stabilization and regulation (performance) point of view, it is desirable to have $\overline{\eta}$ (and η_n) as small as possible.

Since the configuration of the boundedness ellipsoid, characterized by $\overline{P}$ of the boundedness set $\tilde{S}$ of (2.75) and (2.76), changes from one coordinate frame to another, besides η, it is appropriate to consider another parameter (say, ν) as one more rough measure on the size of $\tilde{S}$. Clearly, the radius of the smallest closed ball <u>wholly enclosing</u> the ultimate boundedness set would serve the purpose (complementing η). Accordingly, we define ν

$$\nu \overset{\Delta}{=} [\lambda_{max}(P)/\lambda_{min}(P)]^{1/2}\ \eta \tag{2.79}$$

as the radius of the smallest closed ball wholly enclosing the ultimate boundedness set. Based on this definition and (2.75), (2.76) we also have

$$\overline{\nu} = [\lambda_{max}(\overline{P})/\lambda_{min}(\overline{P})]^{1/2}\ \overline{\eta} \tag{2.80}$$

and

$$\nu_n = [\lambda_{max}(\overline{P})/\lambda_{min}(\overline{P})]^{1/2}\ \eta_n \tag{2.81}$$

Roughly speaking, the parameters η and ν serve as the measures of minor axis and major axis, respectfully, in a two-dimensional (ellipse) setting.

<u>Original Coordinates</u>

$x(t)$

<u>Control Law</u>

$u_o = Gx+p(x)$

G maximizes M*

M.C.I.: β

U.B.M.: η,ν

Transformed Coordinates $\bar{x}(t)$

$\bar{x}(t) = Q^{-1}\, x(t)$

<u>Control Law</u>

$u = \bar{G}^*\bar{x} + \bar{p}(\bar{x})$

$\bar{G}^*$ maximizes $\bar{M}^*(\bar{G}=GQ\neq\bar{G}^*)$

M.C.I.: $\bar{\beta} > \beta$

U.B.M.: $\eta,\ \bar{\nu}$

<u>Original Coordinates</u>

$x(t)$

<u>Control Law</u>

$u = G_n x + p_n(x)$

$G_n = \bar{G}^*\, Q^{-1}$

U.B.M.: $\eta_n,\ \nu_n$

<u>Summary of the Algorithm</u>

<u>Illustrative Example</u>

<u>Example 2.4</u>: Consider

$$\begin{aligned} \dot{x}_1(t) &= r_1(t)\ x_1(t) + [2+r_2(t)]x_2(t) + 2v(t) \\ \dot{x}_2(t) &= -\ 1/2\ x_1(t) + u(t) + v(t) \end{aligned} \tag{2.82}$$

having uncertainties bounded as follows:

$$r(t) = \{[r_1 r_2]^T: r_1^2 + r_2^2 \leq r^2 = \text{constant} > 0$$

$$v(t) \rightarrow V = \{v: |v| \leq \rho_r = \text{constant} > 0$$

We define the nominal pair

$$(A,B) = \begin{bmatrix} 0 & 2 \\ -1/2 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and decompose the uncertain portion as follows:

$$\Delta A(r) = \Delta\tilde{A}(r) = \begin{bmatrix} r_1 & r_2 \\ 0 & 0 \end{bmatrix}; \; C = C_m + \tilde{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Let $G = [0 \quad g_2]$ with $g_2 < 0$.

It turns out that $\max M^*(g_2) = 0.22$, with $g_2 = -2.25$.

$$\text{Now } \tilde{M} = \max_{r_1,r_2} ||\Delta\tilde{A}(r)|| = \max_{r_1,r_2} \sqrt{(r_1^2 + r_2^2)} = \rho_r$$

The mismatch constraint condition is therefore

$$\rho_r < M^*(g_2), \text{ i.e., } \rho_r < 0.22 \text{ and } g_2 = -2.25$$

We now calculate η and ν ($\rho_\nu = \bar{\varepsilon} = 1.0$) and get

ρ_r	0	0.05	0.1	0.15	0.2
η	9.14	11.82	16.72	28.6	99.9
ν	27.72	35.86	50.73	86.77	303.1

Now let us employ the transformation

$$x = Q\bar{x} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \bar{x}$$

The nominal pair will then be $(A,B) = \left(\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and the decomposition is

$$\Delta A(r) = \begin{bmatrix} r_1 & \frac{1}{2} r_2 \\ 0 & 0 \end{bmatrix} = \Delta\tilde{A}(r);\ \bar{C} = C_m + \tilde{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.\quad \text{Now } \bar{G} = [0 \quad \bar{g}_2],$$

and

$$\max_{\bar{g}_2} \bar{M}^* (\bar{g}_2) = 0.3 \text{ with } \bar{g}_2 = -1.57$$

The measure of mismatch is $\tilde{\bar{M}} = \max_{r_1, r_2} \sqrt{r_1^2 + (1/4)r_2^2}$ and thus

$$\tilde{\bar{M}} < \tilde{M} \text{ and } M^* < \bar{M}^* \rightarrow \bar{\beta} > \beta$$

Hence, we can conclude that the ultimate boundedness of (2.82) is assured if $\rho_r < 0.3$ instead of $\rho_r < 0.22$ as given by the analysis before transformation.

Remark 2.5: It is interesting to note that the magnitude of the linear control gain of the final control law (=1.57) is not only less than the control gain originally designed (=2.25), but also tolerates larger uncerainties as evidenced by bigger $\bar{M}^*$. Thus it brings out the point that higher control effort doesn't necessarily mean better robustness (where tolerable uncertainty is the measure of robustness). The calculation of η_n and ν_n gives ($\rho_\nu = \bar{\varepsilon} = 1.0$)

ρ_r	0	0.05	0.1	0.15	0.2	0.25	0.29
η_n	14.2	17	21	28	41	82	399
ν_n	29	35	43	57	85	169	820

Remark 2.6: Comparing the above tables and Figs. 2 & 3, it may be noted η_n is higher than η (recall that both η_n and η are U.B.M's in the original coordinates, η_n being the U.B.M. after the transformation and η, before the transformation) in the range $0 < \rho_r < 0.15$, thereby indicating deterioration in the performance. But it is to be kept in mind that we in turn gained in stability in the sense that the transformation allowed us to get a 'better'

mismatch constraint condition thereby assuring stability and ultimate boundedness for a bigger range of uncertainty ($0.2 < \rho_r < 0.3$). That is η doesn't exist for this range while η_n exists. Thus the typical trade off between stability and performance is brought out by this example. In fact, in this example there is improvement in performance even for the range $0.15 < \rho_r < 0.3$.

Remark 2.7: Since in this example $\tilde{\tilde{M}} < \tilde{M}$ and $M^* < \bar{M}^*$, we achieved reduction in conservatism from both better decomposition as well as better bound points of view. Also note that the particular transformation here may not be the best we can use.

Remark 2.8: It may be noted from this example (and also from [26]) that the gain which maximizes the mismatch threshold M^* also gives the smallest η for the same $\tilde{M}$ (i.e., ρ_r).

FIG 2

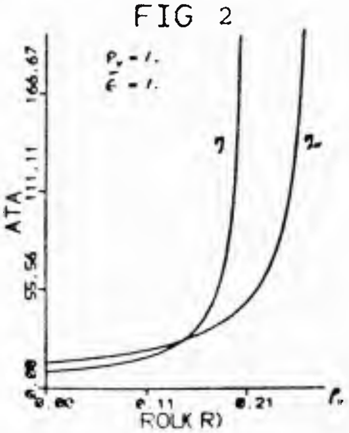
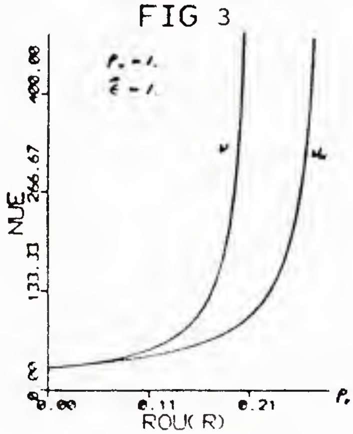


FIG 3



Figures 2 and 3: U.B.M. variation with ρ_r.

2.5 Application of Structured Bound to Stability of Interval Matrices

Another important contribution of the bound developed for structured uncertainty in this research is its applicability to the analysis of 'Interval Matrices', on which there is interesting literature available [30]-[37].

An interval matrix A_I is a real matrix in which all elements are known only to belong to a specific closed interval [30]. An nxn interval matrix A_I is a set of real matrices defined as follows:

$$A_I = [B,C] = \{A = [a_{ij}]: \ a_{ij} \rightarrow [b_{ij}, c_{ij}] \ i,j=1,2,\ldots n\}$$

where

$$B = [b_{ij}], \ C = [c_{ij}] \text{ and } b_{ij} \leqslant a_{ij} \leqslant c_{ij} \tag{2.83b}$$

It is important to note that there is considerable amount of literature available on the stability of interval polynomials starting from Kharitonov [33], Bialas [31], Bose [32], Daoyi [34]. Bialas [31] extends the results of the stability of interval polynomials to the case of interval matrices and presents necessary and sufficient conditions for the stability of interval matrices. But as the counterexamples given by Karl, Greschak and Verghese [36] and Barmish and Hollot [37] indicate, considerable care has to be exercised in extending the results of interval polynomials to the case of interval matrices. Heinen [35] and Daoyi [34] propose simple sufficient conditions for testing the stability of interval matrices. While these methods are indeed simple to use, their application is limited to the case where 'end points' of C, namely c_{ii} of equation (2.83) are negative. In this section we present a different sufficient condition based on the bound μ_Y of (2.6) which does not impose the above requirements.

From the results of Section 2.2, it can be easily concluded that given an asymptotically stable matrix F the interval matrix

$$[F - \varepsilon U_e, \quad F + \varepsilon U_e] \tag{2.84}$$

is stable if ε satisfies the condition

$$\varepsilon < \frac{1}{\sigma_{max}(P_m U_e)_s} \tag{2.85a}$$

where $U_e = [u_{e_{ij}} \overset{\Delta}{=} \varepsilon_{ij}/\varepsilon]$ and P satisfies the Lyapunov equation (2.85b)

$$F^T P + PF + 2I_n = 0 \tag{2.85c}$$

In the following section we extend the above result to establish the stability of the interval matrix A_I of (2.83).

2.5.1 Extension to Stability of Interval Matrices

Let us, as before, denote the matrices A, B, C, etc. as follows

$$A = [a_{ij}], \ B = [b_{ij}], \ C = [c_{ij}] \ \ldots \ i,j = 1,2,\ldots n \tag{2.86}$$

It has been shown by Bialas [31] that the interval matrix $A_I = [B,C]$ is stable only if (necessary condition) the matrices B and C are stable. Thus we start by assuming the matrices B and C to be stable. Then compute the deviations

$$d_{ij} \overset{\Delta}{=} c_{ij} - b_{ij} \text{ for all } i,j = 1,2,\ldots n \tag{2.87}$$

where d_{ij} is seen to be always nonnegative.

Denote $D = [d_{ij}]$ (2.88)

Δ and $d = d_{ij\ max}$ (2.89)

Form $u_{e_{ij}} = d_{ij}/d$ where $U_e = [u_{e_{ij}}]$. (2.90)

We then have the following theorems.

Theorem 2.7: The interval matrix A_I is stable if the matrix B is stable and if

$$d < \frac{1}{\sigma_{max}(P_{1m}U_e)_s} \equiv \mu_{YB} \qquad (2.91)$$

where

$B^T P_1 + P_1 B + 2I_n = 0$ and P_{1m} is the matrix such that $P_{1m_{ij}} \overset{\Delta}{=} |P_{1_{ij}}|$

and U_e is given by (2.90).

Theorem 2.8: The interval matrix A_I is stable if the matric C is stable and if

$$d < \frac{1}{\sigma_{max}(P_{2m}U_e)_s} \equiv \mu_{YC} \qquad (2.92)$$

where

$C^T P_2 + P_2 C + 2I_n = 0$ and P_{2m} is the matrix such that $P_{2m_{ij}} = |P_{2ij}|$ and U_e is given by (2.90).

The above theorems clearly do not require the diagonal elements c_{ii} to be negative in contrast to the case of Heinen [35] and Daoyi [34].

In fact, using the above sufficient condition the interval over which the matrix A_I is stable, can be extended as illustrated by examples that follow.

Finally, since both B and C are stable one can compute both μ_{YB} and μ_{YC} and select the one that satisfies either of the conditions (2.91) or (2.92). If both satisfy the condition ((2.91) or (2.92)), then one can select the maximum of the two so that the interval range can further be expanded.

matrix A_I is stable, can be extended as illustrated by examples that follow.

Example 2.5: Let us consider the same example worked out by Bialas [31] and

$$A_I = [B,C] \text{ where } B = \begin{bmatrix} -5 & 1 \\ 4 & -6 \end{bmatrix}; \; C = \begin{bmatrix} -3 & 2 \\ 5 & -4 \end{bmatrix}$$

For this problem, the condition of Bialas required extensive computation while that of Heinen was not satisfied. Let us now apply the proposed technique to this example.

Clearly, the matrix B is stable. Following equations (2.88), (2.89) and (2.90), the matrices D and U_e are given by

$$D = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and thus } d = 2 \text{ and } U_e = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Solving the Lyapunov equation (2.91), we get

$$P_1 = \begin{bmatrix} 0.27273 & 0.0909 \\ 0.0909 & 0.18182 \end{bmatrix}$$

and

$$\mu_{YB} = 2.0736$$

Since,

$$d = 2 < \mu_{YB} = 2.0736$$

we conclude that the above interval matrix $A_I = [B,C]$ is stable.

In fact, a 'larger' interval matrix that is stable can be obtained for this example, as

$$A'_I = [B',C] \text{ where } B' = \begin{bmatrix} -7 & 0 \\ 3 & -8 \end{bmatrix}; \quad C = \begin{bmatrix} -3 & 2 \\ 5 & -4 \end{bmatrix}$$

where $B' = B-dU_e$ and $C = B+dU_e$.

Example 2.6: Consider

$$A_I = [B,C] \text{ where } B = \begin{bmatrix} -3.2 & -2.2 \\ 0.8 & -0.2 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

Note that $c_{22} = 0$ and thus none of the above methods could be applied to this example. Applying the proposed technique, we have

$$D = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}, \; d = 0.2 \text{ and thus } U_e = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Noting that C is stable and solving the Lyapunov equation (2.92), we get

$$P_2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}$$

and

$$\mu_{YC} = 0.236$$

Since

$$d = 0.2 < \mu_{YC} = 0.236$$

the given interval matrix is stable. A stable interval matrix with expanded range can be formed as

$$A'_I = [B, C'] \text{ where } B = \begin{bmatrix} -3.2 & -2.2 \\ 0.8 & -0.2 \end{bmatrix}; \; C' = \begin{bmatrix} -2.8 & -1.8 \\ 1.2 & 0.2 \end{bmatrix}$$

where

$$B = C - dU_e \text{ and } C' = C + dU_e$$

III. SYNTHESIS OF CONTROLLERS FOR ROBUST STABILITY

The foregoing discussion in Section II is basically concerned with the analysis of stability robustness for linear systems. No effort was made to synthesize a controller to achieve stability robustness except in 2.3.2 where the gain was determined in an ad hoc way for a specific problem. In this section, we address this design aspect from a systematic algorithm point of view. The philosophy behind the proposed procedure is to make use of the perturbation bounds developed in the previous section in a design formulation and give an algorithm to synthesize controllers for robust stability. Towards this direction, a quantitative measure called 'stability robustness index' is introduced and based on this index a design algorithm is presented by which one can pick a controller that possesses good stability robustness property. The algorithm, for given size of perturbation can be used to select the range of control gain for which the system is stability robust or alternatively, for given control gain, can be used to determine the range of the size of allowable perturbations for stability. A VTOL aircraft example [23] is considered in which variations of parameters due to changes in air speed are taken as perturbations and a constant gain linear state feedback control law is presented to attain stability robustness for these perturbations. Various design implications are discussed based on this example.

3.1 Linear State Feedback Control Design Using Perturbation Bound Analysis

As before, consider the linear, time invariant system described by

$$\dot{x} = Ax + Bu \qquad (3.1)$$
$$y = Cx$$

where x is nxl state vector, the control u is mxl and output y (the variables we wish to control) is kxl. The matrix triple (A,B,C) is assumed to be completely controllable and observable. Let the control law be given by $u = Gx$.

For this case, the nominal closed loop system matrix is given by

$$\bar{A} = A + BG \tag{3.2}$$

and G is such that $\bar{A}$ is asymptotically stable.

Let ΔA, ΔB and ΔG be the perturbation matrices formed by the maximum modulus deviations expected in the individual elements of matrices A, B and G respectively. Then for this structured perturbation case, the perturbed system matrix of (3.1) is given by

$$\bar{A}_p = (A + \Delta A) + (B + \Delta B)\ (G + \Delta G) \tag{3.3}$$

<u>Design Observation 3.1</u>: The perturbed system matrix of (3.3) is stable for all perturbations bounded by ΔA, ΔB and ΔG, if

$$\varepsilon = \varepsilon_{ij\ max} = [\Delta A + \Delta B\ G_m + (B + \Delta B)_m\ \Delta G]_{ij\ max} < \frac{1}{\sigma_{max}(P_m U_e)_s} \equiv \mu_y \tag{3.4}$$

where P satisfies $P\bar{A} + \bar{A}^T P + 2I_n = 0$ and $P_{mij} = |P_{ij}|$.

Note that the 'error' matrix in the left hand side (l.h.s.) of (3.4) comprises the perturbation matrices ΔA, ΔB and ΔG as well as nominal matrices G and B. In practice, the following situations may arise.

<u>Case A</u>: The matrices ΔA, ΔB and ΔG are known (of course G_m and B are known too): for this case, the U_e matrix in the right hand side (r.h.s.) is simply formed by

$$U_{eij} = \varepsilon_{ij}/\varepsilon \tag{3.5}$$

<u>Case B</u>: Matrices ΔA, ΔB and ΔG are partially known: in this case one needs at least any two ratios among $(\Delta A_{ij\ max}/\Delta B_{ij\ max})$ or $(\Delta A_{ij\ max}/\Delta G_{ij\ max})$ or $(\Delta B_{ij\ max}/\Delta G_{ij\ max})$ to be able to form the U_e matrix (in which case the U_e matrix contains some unity elements at the appropriate places).

<u>Case C</u>: No information on the perturbation matrices in the l.h.s. is known: for this case, one has to go for the worst case situation and replace U_e by U_n. In other words the stability condition becomes

$$\varepsilon < \frac{1}{\sigma_{max}(P_m U_n)_s} \tag{3.6a}$$

where $E = [\Delta A + \Delta B\, G_m + (B + \Delta B)_m\, \Delta G]$. (3.6b)

It may be seen that similar development applies to the 'unstructured' perturbation case also and hence the details are not given here.

3.1.1 Stability Robustness Index and Design Algorithm

A first glance at the stability condition (3.4), may suggest that the bound μ_Y (for structured perturbations) can serve as a measure of stability robustness and that higher this bound, better is the stability robustness. It may thus be concluded that the control gain G which makes μ_Y maximum is the gain to look for. However a closer look at the condition reveals that both l.h.s. and r.h.s. terms of the conditions (3.4) are functions of the control gain G and a variety of control gains G may satisfy the proposed stability condition for given perturbations. It is thus more appropriate to consider the relative difference of these two terms in comparing and synthesizing different controllers from stability robustness point of view. To this end we define, as a measure of stability robustness, an index called 'Stability Robustness Index' given by

$$\beta_{S.R.} \triangleq (\mu_Y - \varepsilon) \quad \text{(for structured perturbations)} \tag{3.7a}$$

for Case A situation and

$$\beta_{S.R.} \triangleq \mu_Y \quad \text{(structured perturbation)} \tag{3.7b}$$

for Cases B and C situations described before.

By this definition, for Case A, $\beta_{S.R.} > 0$ corresponds to the stability robustness region.

Even though the 'stability robustness index' defined above plays the role of a 'stability margin', it is quite different in its interpretation from the standard notions of 'stability margin'. One classical relative stability

margin in Single Input Single Output (SISO) stable systems is the distance, α, of the nearest characteristic root from the imaginary axis. Another margin of stability in SISO systems is the standard gain/phase margin, namely the gain/phase changes that can be tolerated before the system becomes unstable. There are, of course, investigations being carried out to generalize these notions for multivariable systems [1]. In contrast to these interpretations, $\beta_{S.R.}$ is a stability margin given in terms of the tolerable system parameter perturbations. That is, higher the index $\beta_{S.R.}$, higher is the tolerable parameter perturbation for stability. Clearly these different notions of stability margins do not imply one another, except in special circumstances. Thus higher $\beta_{S.R.}$ does not necessarily mean higher α and vice versa. It may also be noted from (3.4) that for linear systems when $\Delta A = 0$, $\Delta B = 0$ and $\Delta G \neq 0$ (unknown), $\beta_{S.R.}$ $(=\mu_Y)$ may be regarded as a 'gain margin', albeit in a restricted sense.

3.1.2 Control Design Algorithm for Robust Stability

It is clear from the expressions for μ_Y (3.4), the 'error matrix' (3.6b) and $\beta_{S.R.}$ (3.7) that these quantities depend on the control gain G and as gain G is varied $\beta_{S.R.}$ changes. In order to plot the relationship between $\beta_{S.R.}$ and the gain G, we need a scalar quantitative measure of G. For this we can either use

$$J_{cn} = ||G||_s = \sigma_{max}(G) \tag{3.8a}$$

or

$$J_{cn} = [\int_0^\infty (u^T u)dt]^{1/2} = [\int_0^\infty x^T G^T G x dt]^{1/2} \tag{3.8b}$$

where J_{cn} denotes a measure of 'nominal control effort'.

The variation of $\beta_{S.R.}$ with the control effort J_{cn} is very much dependent on the perturbation matrices and on the behavior of the Lyapunov solution,

which cannot be described analytically in a straightforward way. Assuming stability robustness is the only design objective, the design algorithm basically consists of picking a control gain that maximizes stability robustness ($\beta_{S.R.}$). Specifically the algorithm involves determining the index $\beta_{S.R.}$ and the control effort J_{cn} for different values of the control gain G and plotting these curves. These design curves can then be used to pick a gain that achieves a high $\beta_{S.R.}$. The algorithm thus provides a simple constant gain state feedback control law that is robust from stability point of view. The algorithm, for given perturbations, can be used for selecting the range of control effort for which the system is stability robust or alternatively for given control effort, can be used to determine the range of allowable perturbations for stability.

In the next section a widely used VTOL aircraft control problem with varying flight conditions is considered and a stability robust controller is synthesized using the proposed methodology.

3.1.3 Robust Control Design for VTOL Aircraft

In this section, we recall the VTOL aircraft (helicopter) problem described in section (2.3).

The linearized model of the VTOL aircraft in the vertical plane is described by

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u, \quad x(0) = x_o \tag{3.9}$$

The components of the state vector $x \varepsilon R^4$ and the control vector $u \varepsilon R^2$ are given by

$x_1 \rightarrow$ horizontal velocity (knots)

$x_2 \rightarrow$ vertical velocity (knots)

$x_3 \rightarrow$ pitch rate (degree/s)

$x_4 \rightarrow$ pitch angle (degrees)

$u_1 \rightarrow$ 'collective' pitch control

$u_2 \rightarrow$ 'longitudinal cyclic' pitch control

Essentially, control is achieved by varying the angle of attack with respect to air of the rotor blades. The collective control u_1 is mainly used for controlling the motion of the aircraft vertically up and down. Control u_2 is basically used to control the horizontal velocity of the helicopter.

In [23], the mathematical model is presented assuming the nominal airspeed to be 135 knots. For this nominal case (ΔA and $\Delta B = 0$) the matrices A and B are given by

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.10a}$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix} \tag{3.10b}$$

We take the initial condition to be

$$x^T(0) = [0.85 \quad 0.15 \quad 0 \quad -0.05] \tag{3.11}$$

As the airspeed changes significant changes take place in the elements a_{32}, a_{34} and b_{21}. Let us consider, for illustration purposes, the following range of variations in these parameters.

Case 1: $0.3545 \leq \bar{a}_{32} = 0.3681 \leq 0.3817$

$1.31 \leq \bar{a}_{34} = 1.42 \leq 1.53$

$3.39 \leq \bar{b}_{21} = 3.544 \leq 3.702$

i.e., $|\Delta a_{32}| = 0.0136; \ |\Delta a_{34}| = 0.11; \ |\Delta b_{21}| = 0.157$ (3.12)

Case 2: $|\Delta a_{32}| = 0.041; \ |\Delta a_{34}| = 0.332; \ |\Delta b_{21}| = 0.47$ (3.13)

Case 3: $|\Delta a_{32}| = 0.068; \ |\Delta a_{34}| = 0.553; \ |\Delta b_{21}| = 0.78$ (3.14)

Case 4: $|\Delta a_{32}| = 0.095; \ |\Delta a_{34}| = 0.774; \ |\Delta b_{21}| = 1.097$ (3.15)

Case 5: $|\Delta a_{32}| = 0.1363; \ |\Delta a_{34}| = 1.106; \ |\Delta b_{21}| = 1.5674$ (3.16)

In other words, the matrices ΔA and ΔB are known for these five cases.

For designing the nominal state feedback control that stabilizes the nominal closed loop system, for this example, we employ the standard linear quadratic optimal control design algorithm assuming a performance index

$$J = \int_o^\infty (x^T Q x + \rho_c u^T R_o u)\,dt$$

where Q and R_o are (nxn), and (mxm) symmetric positive definite matrices and ρ_c is a scalar variable used for designing the control gain G.

For this case, the nominal closed loop system matrix is given by

$$\bar{A} = A + BG, \quad G = -R_o^{-1} B^T K/\rho_c \tag{3.17}$$

and

$$KA + A^T K - KB \frac{R_o^{-1}}{\rho_c} B^T K + C^T Q\, C = 0 \tag{3.18}$$

and $\bar{A}$ is asymptotically stable. For each case using

$$Q = \begin{bmatrix} 0.04 & 0.01 & 0 & 0.01 \\ 0.01 & 0.025 & 0.01 & 0 \\ 0 & 0.01 & 0.01 & 0 \\ 0.01 & 0 & 0 & 0.01 \end{bmatrix}$$ and $R_o = I_2$ and ρ_c as design variable, the standard optimal LQ regulator control gain and the corresponding control effort J_{cn} of (3.8b) are computed and $\beta_{S.R.}$ is calculated. The plots of $\beta_{S.R.}$ vs. J_{cn} for these five cases are shown in Fig. 4.

In interpreting these plots, it is to be recalled that the region of control effort for stability robustness is the region in which $\beta_{S.R.} > 0$.

Design observation 3.1: From these plots, it may be observed that as the parameter perturbation range is increased, the range of control effort for stability robustness decreases.

Design observation 3.2: For given set of parameter perturbations, there is a unique control effort for which $\beta_{S.R.}$ is maximum. Obviously this is the control effort (and the corresponding control gain) that we seek, assuming practical constraints are satisfied.

Design observation 3.3: It is to be noted that for all these cases, the maximum $\beta_{S.R.}$ occurs almost at the same control effort. However it is also to be kept in mind that the range of variations considered in cases 2 through 5 are simply some multiples of the range of case 1.

Now let us consider the cases when there are parameter perturbation in only one of the matrices, A or B. Accordingly, we consider the following ranges.

Case 6: $|\Delta a_{32}| = 0.3018$; $|\Delta a_{34}| = 1.300$; $|\Delta b_{21}| = 0$

Case 7: $|\Delta a_{32}| = 0.1366$; $|\Delta a_{34}| = 1.106$; $|\Delta b_{21}| = 0$

Case 8: $|\Delta a_{32}| = 0$; $|\Delta a_{34}| = 0$; $|\Delta b_{21}| = 2.5671$

Case 9: $|\Delta a_{32}| = 0$; $|\Delta a_{34}| = 0$; $|\Delta b_{21}| = 1.5674$

Figure 5 corresponds to these cases. It may be observed from these plots that, as before, it turns out that smaller the size of the perturbation, the more is the control effort range. But consider cases 6 and 9. From plots corresponding to these two cases, it can be seen that the control range (for stability) for perturbations in A is larger than the control

range for perturbations in B indicating variations in matrix B are more critical from stability robustness point of view.

Extention to LQG Regulators and Dynamic Compensator Design

Clearly the proposed methodology can be extended to Linear Quadratic Gaussian (LQG) regulators and dynamic compensators in a straightforward way. However, the design implications in this case could be different. This extension is suggested as a future research topic.

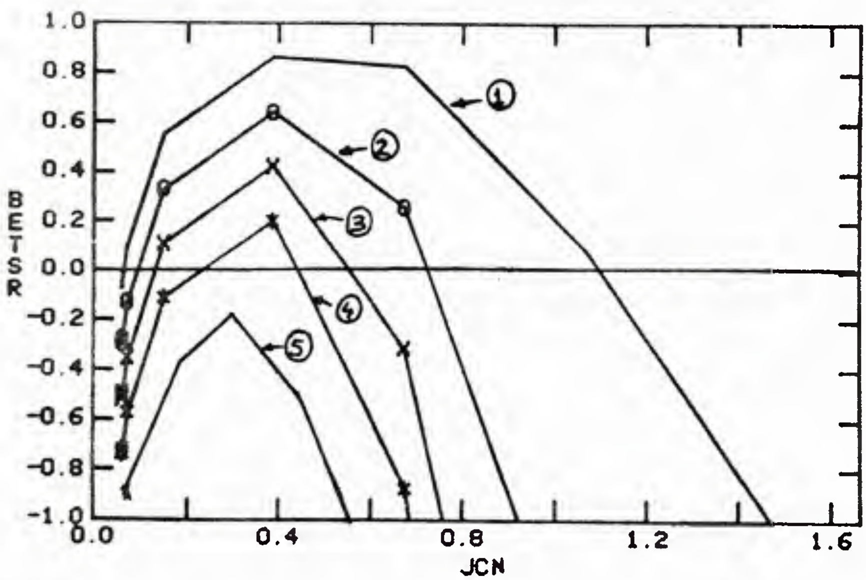


Figure 4: Variation of β_{SR} with nominal control effort $J_{cn}(\Delta A \neq 0, \Delta B \neq 0)$.

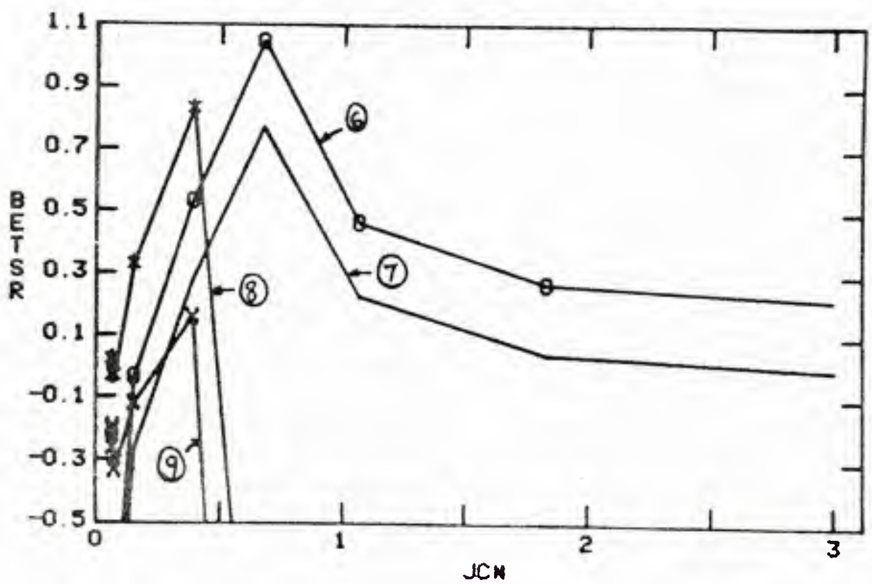


Figure 5: Variation of β_{SR} with nominal control effort $J_{cn}(\Delta A = 0, \text{ or } \Delta B = 0)$

IV. ANALYSIS FOR PERFORMANCE ROBUSTNESS OF LINEAR REGULATORS

At this stage, it is instructive to pause for a moment and recapitulate the material that has been discussed in the previous sections. The basic aspect addressed was that of 'stability robustness' namely the assumption that the fundamental 'acceptable' behavior is the stability of the closed loop system in the presence of perturbations. However, in many practical situations, guaranteeing stability alone may not be sufficient to carry out the mission objective. This is particularly so in the 'linear regulator' problems with stringent requirements on the 'regulation' aspect, especially in aerospace applications.

It is the purpose of the control design to 'regulate' the controlled variables (or state) and the performance of the regulator is deemed 'acceptable' if the 'closed loop regulation cost' in the presence of these perturbations is within a bound specified by the designer or the design specifications. Thus, not only stability but a specified degree of performance (regulation) are required. This aspect is termed the 'Performance (Regulation) Robustness' and this section addresses the aspect of analysis and design of linear quadratic regulators for this kind of 'performance robustness'.

4.1 Brief Review of Literature

The 'robustness' of linear multivariable regulators has been analyzed by many researchers. Anderson and Moore [38] have shown that the single input Linear Quadratic State Feedback (LQSF) designs have a phase margin of 60° and infinite gain margin. Recent studies by Wong and Athans [39] and Safonov and Athans [40] conclude that the multi input LQ regulators possess a guaranteed gain margin - 6 db to $+\infty$db and a phase margin of +60° in all channels. It is shown by them that no such guaranteed margins can be given

for Linear Quadratic Gaussian (LQG) regulators. The robustness results presented in these above references are primarily concerned with allowable perturbation in the feedback gains. However these robustness properties cannot directly be extended to the case of perturbations in system matrices (i.e., parameter variations). In fact, it is shown recently by Soroka and Shaked [41] that despite possessing attractive gain and phase margin properties, the LQ regulator suffers from poor stability robustness from parameter variations point of view. It is for this reason that we focus our attention on the analysis and design for robustness with respect to parameter variations.

Ly and Cannon [42] give stochastic representation for parameter uncertainty whereas this report considers the uncertain parameters to vary within certain bounds. The work of Leitmann [24], Barmish and colleagues [26] on ultimate boundedness control of mismatched uncertain system addresses the core problem of regulation but the control law given is nonlinear while the aim of this section is to utilize the simplicity of linear feedback control law of the standard optimal LQ regulators. Lunze [6], using comparison systems concept and Owens and Chotai [7] by combining frequency domain and time domain treatments, present attractive schemes involving more general perturbations. Their methods do not specifically address regulators with quadratic performance indices. Desoer et al [8] have established conditions for stability robustness of linear multivariable interconnected systems for sufficiently small perturbations.

But these methods do not specifically address the type of 'regulation robustness' outlined above. In the context of regulation robustness, Fujii and Mizushima [43] address the robustness of the optimality property of linear regulators but the analysis is restricted to small parameter variations and is carried out in frequency domain. Rolnik and Horowitz [44] present a

method suitable for large parameter variations, again in frequency domain, but for a third order system. In time domain, interesting work is reported on the robustness of linear regulators by Sezer and Siljak [45,46,47] using the concept of suboptimality index which is used as a measure for system robustness both with respect to parameter variations as well as suboptimal control law. The analysis is done for time invariant perturbations in the context of interconnected systems. Krishnan and Brzezowski [48] present an iterative design algorithm such that prescribed trajectory insensitivity is achieved but the method is aimed at time invariant perturbations. Barnett and Story [49] consider a specific variation for which the perturbed cost is same as the nominal optimal cost. McClamroch et al [50] offer a method suitable only for small number of uncertain parameters and fow low order systems. An interesting method by Burghardt [51] proposes the weighting matrices to be functions of the uncertain parameters. The method is ad-hoc and iterative in nature. The method of Evans and Xyaoni [52] considers time invariant perturbations and employs parameter optimization techniques to design the controller.

From a different perspective, there is interesting literature on the analysis and design procedures for tolerable perturbations (Perturbation Bound Analysis) for robust stability and regulation. References [9]-[15] deal with bounds for robust stability alone with no concern for the regulation robustness. Rissanen [53] presents bounds on the linear perturbation of the nominal regulator such that the perturbed regulation cost is below the nominal cost times a given constant greater than 1. The proposed bound holds good for time varying perturbations since the method uses Lyapunov approach but no synthesis procedure is given. Sarma and Deekshatulu [54] address a similar problem but the method is suitable only for small number of uncertain parameters and low order systems. Perhaps, one of the best known <u>design</u> methods

in time domain for time varying perturbations is the method of Guaranteed Cost Control of Chang and Peng [13]. (Ref. [55] by Vinkler and Wood extends this method for time invariant perturbations.) The design procedure proposed in this paper belongs, in principle, to the framework of Guaranteed Cost Control method in the sense that it is a design method in time domain suitable for time varying perturbations, belonging to bounded intervals. The relative merits of these methods are discussed later. The proposed procedure basically formulates the available Perturbation Bound Analysis into an algorithm to design controllers for regulation robustness. Towards this direction, in this section, we present perturbation bounds for 'robust regulation' and then in Section V present a design algorithm to synthesize controllers for robust regulation.

4.2 Perturbation Bounds for Robust Regulation

In this section, we briefly review the upper bounds for robust stability, presented in previous sections (for structured uncertainty).

Upper Bounds for Robust Stability: Consider $\dot{x} = Ax$ (4.1)

$$\dot{X} = [A + E(t)]X \tag{4.2}$$

The system of (4.1) is stable if

$$\varepsilon < \frac{\sigma_{min}(Q)}{\sigma_{max}(P_m U_e)_s} = \mu_y \tag{4.3a}$$

where P satisfies the equation

$$PA + A^T P + 2Q = 0 \tag{4.3b}$$

and U_e is an nxn matrix whose entries are such that

$$U_{eij} = \varepsilon_{ij}/\varepsilon \qquad (0 \leq U_{eij} \leq 1)$$

Note that U_e (and thus μ_y) can be computed even if one knows only the ratio $\varepsilon_{ij}/\varepsilon$ instead of knowing ε_{ij} (and ε) separately.

One important difference in the formula for μ_y of (3.4) above and the bound presented in (4.3) is the presence of $\sigma_{min}(Q)$ in the numerator of (4.3). However, μ_y of (4.3) is the more general expression (Refs. [13]-[53]) and μ_y of (3.4) is simply a special case of (4.3) where $Q = I$. The reason we now use μ_y of (4.3) is the fact that in this section we intend to relate this bound for robust stability to the bound we are going to present for robust regulation. When only robust stability is the primary concern (with no concern for robust regulation) then, of course, we can use μ_y of (3.4) since $Q = I$ maximizes this bound as shown in [14].

We now define precisely our interpretation of 'robust regulation' and then present perturbation bounds to achieve this kind of 'robust regulation'.

Upper Bounds for Robust Regulation

Let

$$J_n \triangleq \int_o^\infty x^T Q \, x \, dt \tag{4.4}$$

$$J_p \triangleq \int_o^\infty X^T Q \, X dt \tag{4.5}$$

We deem the performance of the above system to be 'acceptable' if

$$J_p < J_m \tag{4.6}$$

where J_m is the 'maximum tolerable regulation cost'. Suppose

$$J_m = k \, J_n$$

Definition: The system of (4.2) is 'robust' from performance (regulation) point of view if

$$J_p < k \, J_n (= J_m) \tag{4.8}$$

for some given $k > 1$.

With this definition, the bounds presented by Rissanen [53] on E can be extended as follows.

i) For unstructured perturbations: For system of (4.2), the performance robustness criterion of (4.8) is satisfied if

$$\sigma_{max}(E(t)) \leq \frac{k-1}{k} \frac{\sigma_{min}(Q)}{\sigma_{max}(P)} \tag{4.9}$$

where P satisfies the equation (4.3b).

ii) For structured perturbation: For system of (4.2), the performance robustness criterion of (4.8) is satisfied if

$$\varepsilon \leq \frac{k-1}{k} \frac{\sigma_{min}(Q)}{\sigma_{max}(P_m U_e)_s} \tag{4.10a}$$

$$\leq \frac{k-1}{k} \mu_y = (1 - \frac{1}{k}) \mu_y \tag{4.10b}$$

Henceforth, the discussion will be focused on 'structured' perturbation case, keeping in mind that similar development applies to the 'unstructured' perturbation case also.

From equations (4.3), (4.10) it is clear that the perturbation bound for 'robust regulation' is always less than or equal to the bound for 'robust stability'.

Extension to LQ Regulators

Consider the linear, time invariant system described by

$$\dot{x} = Ax + Bu \tag{4.11}$$

$$y = Cx \qquad \text{(C is full rank)} \tag{4.12}$$

where x is nxl state vector, the control us is mxl and output y (the variables we wish to control) is $\bar{k}$xl. Let the performance index for the above system be given by

$$J = \int_o^\infty (y^T Q y + \rho_c u^T R_o u)dt \tag{4.13}$$

where Q and R_o are ($\bar{k}$x$\bar{k}$) and (mxm) symmetric positive definite matrices, respectively, and ρ_c is a scalar variable used for design. The matrix triple (A,B,C) is assumed to be completely controllable and observable.

For this case, the nominal closed loop system matrix is given by

$$\dot{x} = \bar{A}x \tag{4.14a}$$

$$\bar{A} = A + BG, \quad G = -R_o^{-1} B^T K/\ _c \tag{4.14b}$$

and

$$KA + A^T K - KB \frac{R_o^{-1}}{\rho_c} B^T K + C^T Q C = 0 \tag{4.14c}$$

and $\bar{A}$ is asymptotically stable.

Let ΔA, ΔB and ΔG be the perturbation matrices formed by the maximum modulus deviations expected in the individual elements of matrices A, B and G, respectively. Then for this structured perturbation case, the perturbed system matrix is given by

$$\dot{X} = \bar{A}_p X \tag{4.15a}$$

where

$$\bar{A}_p = (A + \Delta A) + (B + \Delta B)\ (G + \Delta G) \tag{4.15b}$$

The basic premise of 'Regulation Robustness' involves the definition of 'acceptable' performance. In linear regulator problems, it is the purpose of the control design to 'regulate' the state and the efficacy of the design and is measured by the 'closed loop regulation cost' defined by

$$J_{xn} \triangleq \int_o^\infty x^T \bar{Q} x \, dt. \quad \bar{Q} = C^T Q C \tag{4.16}$$

subject to equation (4.1). But in the presence of perturbations, with the fixed controller of (4.14), the 'perturbed regulation cost' defined by

$$J_{xp} \triangleq \int_o^\infty X^T \bar{Q} X \, dt$$

subject to equation (4.15), may deviate considerably from J_{xn} of (4.16). However, the design specifications may dictate that the performance is 'unacceptable' if the 'closed loop regulation cost' exceeds a specific given value. Thus, we deem the performance of the regulator to be 'acceptable' if $J_{xp} < J_m$ (of course $J_{xn} < J_m$) where J_m is the 'maximum tolerable regulation cost' (assuming the control effort to achieve this is acceptable).

As before, letting

$$J_m - k\, J_{xn}, \quad k > 1 \tag{4.18}$$

we say, the linear quadratic regulator is 'robust' from performance (regulation) point of view if

$$J_{xp} < k\, J_{xn} \tag{4.19}$$

for some $k > 1$.

<u>Design observation 4.1a:</u> The perturbed LQ Regulator system matrix of (4.15) is stable for all perturbations bounded by A, B, and G if

$$E \stackrel{\Delta}{=} [\Delta A + \Delta B\, G_m + (B + \Delta B)_m\, G]_{ij\ max}$$

$$< \frac{\sigma_{min}(C^T Q C)}{\sigma_{max}(P_m U_e)_s} \equiv \mu_y \tag{4.20a}$$

where P satisfies $P\bar{A} + \bar{A}^T P + 2C^T Q C = 0$ (4.20b)

<u>Design observation 4.1b:</u> The perturbed LQ Regulator satisfies the regulation robustness criterion of $J_{xp} < k\, J_{xn} = J_m$ if

$$\varepsilon < \frac{k-1}{k}\, \mu_y \equiv \mu_{yp} \tag{4.21}$$

where ε and μ_y are as defined in (4.20).

Having developed the perturbation bounds, we now present a design algorithm, using these bounds, to synthesize controllers for robust regulation.

V. SYNTHESIS OF CONTROLLERS FOR ROBUST REGULATION

In Section III, a synthesis procedure for linear systems to have stability robustness was presented using the perturbation bounds for robust stability. In that discussion, it was assumed that the main design objective was stability alone with no concern for regulation. In this section, we use essentially the same principle to synthesize controllers for linear regulators for maintaining not only stability but an 'acceptable' level of performance (or regulation). Even though the concept is similar, the design implications are somewhat different in this proposed algorithm.

Introducing a quantitative measure called 'Performance Robustness Index' a design algorithm is presented by which one can achieve a trade off between nominal performance and regulation robustness. The proposed method is illustrated with the help of a simple example and results are discussed.

5.1 Performance Robustness Index and Design Algorithm

From (4.18) and (4.21), it can be seen that the performance robustness criterion involves two quantities, namely J_m, the tolerable regulation cost and μ_{yp}, the tolerable perturbation. The tolerable regulation cost J_m can be looked at from two viewpoints: either i) it can be specified as a constant number in which case the index k and the nominal regulation cost J_{xn} are such that their product yields the specified constant number, or ii) it can be given in terms of a constant k times the nominal regulation cost J_{xn}. In the former case J_m is constant whereas in the latter case J_m is an explicit function of J_{xn}. In either case it may be noted that both the left hand side (l.h.s.) terms of the condition are functions of the control gain G and a variety of control gains may satisfy the proposed performance robustness condition for

given perturbations and J_m. Hence, to compare and synthesize different controllers from robustness point of view, there is a need for defining a 'measure' of performance (regulation) robustness. From (4.18), (4.21) it is clear that regulation robustness may be measured either in terms of the perturbation bound (μ_{yp}) or regulation bound (J_m), depending upon the specifications. Accordingly, we define a generic index called 'Performance Robustness Index, $\beta_{P.R.}$' as a measure of performance (regulation) robustness. In what follows, we delineate the different situations one encounters in a design environment and the corresponding interpretation for $\beta_{P.R.}$. For each case a design algorithm is presented based on this index.

Case ia: J_m (constant) is given (i.e. k is known) and ε is known a priori (i.e. ΔA, ΔB ... are known).

Case ib: J_m (variable) is known (i.e. constant k is given) and ε is known a priori (i.e. ΔA, ΔB ... are known).

Since in these two cases, the tolerable regulation J_m is specified, we take the tolerable perturbation μ_{yp} as the design objective. Accordingly, one appropriate measure of performance robustness is the relative size of the residual perturbation. Thus we define

$$\beta_{P.R.} = (\mu_{yp} - \varepsilon) \tag{5.1}$$

Since ε is presumed known, one has to check whether the given perturbation satisfies the condition (4.21) or not. As ε and μ_{yp} are both functions of the control gain G, the performance robustness region is given by $\beta_{P.R.} > 0$.

Design Algorithm: For this case, the design algorithm is as follows:

1) No specific requirements on the nominal performance (i.e. on J_{xn}):

For this situation, the algorithm basically consists of picking a control gain (i.e. ρ_c) that maximizes $\beta_{P.R.}$ (= μ_{yp}-ε) in the region $\beta_{P.R.} > 0$.

2) Concern for both nominal performance as well as performance robustness:

For this situation, the algorithm consists of plotting $\beta_{P.R.}$ vs. the nominal control effort Jcn $|(\overset{\Delta}{=} \int_0^\infty u^T u \, dt)^{1/2}$ and J_{xn} vs. J_{cn} and picking a control gain that achieves a reasonable trade off between these variables.

Note that these design curves can be used, for given set of perturbations, to select the range of nominal control effort for which the system is performance robust.

Case ii(a): J_m (constant) is given (i.e. k is known) and ε is not known.

Case ii(b): J_m (variable) is known (i.e. constant k is given) and ε is not known.

This case is essentially the same as before except that, this time we define

$$\beta_{P.R.} = \mu_{yp} \tag{5.2}$$

In other words, tolerable perturbation is still taken as the design objective.

Design Algorithm: Essentially the same as before except that we do not need the requirement of checking whether $\beta_{P.R.}$ is positive or not (it is always positive).

Note that for this cases, the design curves can be used, for given control effort, to determine the range of size of allowable perturbation.

For case (i) and (ii), the performance robustness index $\beta_{P.R.}$ is in terms of tolerable perturbation bound and thus higher $\beta_{P.R.}$ indicates better performance robustness.

Case iii(a): J_m is now known (i.e. k is not given and ε is known).

Since, in this case, the tolerable regulation cost J_m (and thus k) is not given, we will use it as a design objective. In other words our aim would be to minimize J_m under the worst case situation of the given perturbation being

equal to the bound μ_{yp}. Accordingly, from the condition (4.21), we write

$$\varepsilon = (1 - \frac{1}{k})\ \mu_y \equiv \mu_{yp} \tag{5.3}$$

from which we get

$$k = \frac{\mu_y}{\mu_y - \varepsilon} \tag{5.4}$$

We now define the performance robustness index $\beta_{P.R.}$ as

$$\beta_{P.R.} \overset{\Delta}{=} 1/J_m = 1/(k\ J_{xn}) \tag{5.5}$$

where k is given by (5.4) and J_m is required to be positive. With this definition, $\beta_{P.R.}$ is expressed in terms of the tolerable regulation bound (rather than the perturbation bound as in cases i and ii). The above definition is such that lower the value of J_m, better (or higher) is the performance robustness.

Evidently, for this situation one would like to achieve a low value of J_m (>0) from high performnce robustness point of view and a low value of J_{xn} from nominal performance point of view (assuming adequate control effort is available). It is important to note that μ_y, ε and J_{xn} are all functions of the control gain G (or the control weighting factor ρ_c) and their variation with G (or ρ_c) is very much dependent on the perturbation matrices and on the behavior of the Lyapunov solution which cannot be ascertained analytically or qualitatively in a straightforward manner, although it may be possible to make general comments in some special cases of perturbation. Hence in some problems it may be possible to achieve both high $\beta_{P.R.}$ (low J_m) as well as low J_{xn} with the same control gain whereas in some cases, a trade off between $\beta_{P.R.}$ and J_{xn} may be necessary.

Design Algorithm: For this case, the design algorithm consists of plotting $\beta_{P.R.}$ vs. the nominal control effort (J_{cn}) and picking a gain that maximizes

$\beta_{P.R.}$ (in the case when no specific requirement on J_{xn} exists) or picking a gain that achieves a reasonable trade off between $\beta_{P.R.}$ and J_{xn}.

Case iii(b): J_m is not specified (i.e. no concern for regulation robustness).

For this situation, we basically design for robust stability as discussed in Section III.

5.2 Application Example

Consider the simple scalar system

$$\dot{x} = -x + u, \quad x_o = 1 \tag{5.6}$$

$$J = \min_{u} \int_{o}^{\infty} (x^2 + \rho_c u^2)dt \tag{5.7}$$

Case 1: J_m is given and ε is given. Let $k = 3$ so that $J_m = 3J_{xn}$ and let

$$|\Delta a| = 0.5 \text{ and } |\Delta b| = 1.207 \tag{5.8}$$

Then the calculation of ε, μ_{yp}, J_{xn} and J_m is summarized in Table 4.

From this it is seen that ε is greater than μ_{yp} but less than μ_y in the range of ρ_c considered. Thus it can be concluded that one can guarantee stability for the given perturbtions but cannot guarantee $J_{xp} < kJ_{xn} = 3J_{xn}$. Either the range of perturbations Δa, Δb have to be smaller to be able to guarantee $J_{xp} \leq 3J_{xn}$ or, for the given Δa, Δb, one has to sacrifice more in performance by allowing higher k (i.e. J_m). For example if k is increased from 3 to 37, the resulting μ_{yp} and J_m are as shown in Table 5.

For this situation, it is seen that higher the control effort (lower the ρ_c) better is the performance robustness in the sense that higher is the perturbation bound for robust regulation ($J_{xp} < 37J_{xn}$).

Case ii(b): J_m is given and ε is not given. Let $J_m = J_o$ (5.9a)

where J_o = open loop regulation cost = $\int_o^\infty x^2\, dt$ (5.9b)

subject $\dot{x} = -x$ (5.10)

Let $J_o = kJ_{xn}$, where

$$J_{xn} = \int_o^{\infty} x^2 \, dt \text{ subject to (5.10).}$$

Thus,

$$k(\rho_c) = \frac{J_o}{J_{xn}(\rho_c)} \qquad (5.11)$$

indicating that k and J_{xn} are functions of the control weighting ρ_c.

The table of the perturbation bound μ_{yp} for robust regulation is given by Table 6.

Thus, for this situation also it is seen that the higher the control effort (lower the ρ_c) better is the performance robustness in the sense that higher is the perturbation bound for robust regulation ($J_{xp} < J_o$).

Case iii(a): J_m is not known (k is not given) and ε is given (and regulation robustness is desired).

As before let $|\Delta a| = 0.5$, $|\Delta b| = 1.207$. This time J_m is the design objective variable with ε being equated to μ_{yp}. Table 7 summarizes the different quantities.

Design Observation 5.1: As anticipated, lesser the ε, closer is k to 1.

Design Observation 5.2: Comparing the entries ε and J_m with those of Table 4, it is seen that a higher J_m is to be tolerated to accommodate the given perturbation Δa and Δb of (5.8).

Design Observation 5.3: It may be noted that in this case, nominal performance (J_{xn}) and performance robustness ($\beta_{P.R.}$) have opposing trends indicating the need for a trade off. A reasonable choice for a 'robust' control gain could be the gain corresponding to $\rho_c = \frac{1}{2}$ or $\frac{3}{4}$ or 1. However, if there is no stringent requirement on the nominal performance then the control gain corresponding to

higher ρ_c (such as 2 above) can be selected since that gain achieves a low value for J_m for the given perturbation.

TABLE 4

K=3

ρc	1/8	1/4	1/2	3/4	1	5/4	3/2	2
ε	2.914	1.992	1.38	1.13	1	0.912	0.85	0.77
μ_{yp}	2	1.236	0.732	0.521	0.414	0.3416	0.29	0.224
J_m	0.5	0.6708	0.865	0.982	1.06	1.118	1.162	1.225
μ_y	3	2.236	1.732	1.523	1.414	1.341	1.29	1.224

TABLE 5

ρc	1/8	1/4	1/2	3/4	1	5/4	3/2	2
ε	2.914	1.992	1.38	1.13	1	0.912	0.85	0.77
μ_{yp}	2.918	2.175	1.685	1.481	1.375	1.305	1.255	1.19
J_m	6.18	8.273	10.68	12.11	13.08	13.79	14.33	15.11
J_{xn}	0.167	0.2236	0.2886	0.3273	0.3535	0.3726	0.3875	.4085

TABLE 6

ρc	1/8	1/4	1/2	3/4	1	5/4	3/2	2
k	3	2.36	1.732	1.527	1.4149	1.3416	1.29	1.224
μ_{yp}	2	1.236	0.732	0.527	0.414	0.3416	0.29	0.224
$J_m=J_o$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
μ_y	3	2.236	1.732	1.5227	1.414	1.3416	1.29	1.224
J_{xn}	0.167	0.2236	0.2886	0.3273	0.3535	0.37265	0.3875	0.4085

TABLE 7

ρ_c	1/8	1/4	1/2	3/4	1	5/4	3/2	2
ε	2.914	1.992	1.38	1.13	1	0.912	0.85	0.77
μ_y	3	2.236	1.732	1.525	1.414	1.34	1.29	1.224
$k=\frac{\mu_y}{\mu_y-\varepsilon}$	34.84	9.174	4.926	3.86	3.42	3.134	2.932	2.695
$J_m=kJ_{xn}$	5.818	2.05	1.42	1.26	1.21	1.16	1.13	1.098
J_{xn}	0.167	0.2236	0.289	0.327	0.3535	0.3726	0.3875	0.408
$\beta_{P.R.}=\frac{1}{J_m}$	0.172	0.488	0.704	0.793	0.826	0.862	0.885	0.91

VI. EXTENSION TO LARGE SPACE STRUCTURE MODELS

Up to this point, the analytical development to analyze and synthesize robust controllers using perturbation bounds was presented. Some multiple input multiple output system examples have been presented to illustrate the concepts. One application that is of interest to the present research under this contract is that of the control of Large Space Structures (LSS). It is clear that the proposed development can be easily illustrated using an LSS model data. However, the main purpose of this section is to be able to exploit the special structure of LSS models and specialize the results obtained so far to the case of LSS models. Towards this direction, in section 6.1 we obtain explicit expressions for the perturbation bounds μ, treating parameter variation as perturbation, in terms of modal data and draw some conclusions about their usefulness. Then in section 6.2, the mode truncation problem is treated as an additive perturbation and the results of section 6.1 are applied to this case which results in a preliminary algorithm to determine the magnitude of the control gain apriori as a function of the number of modes one controls.

6.1 Uncertain Modal Data as Perturbation

It is known that the typical differential equation describing the dynamics of 'modal coordinates' of an LSS model, assuming no control inputs (i.e. open loop case), is given by

$$\ddot{\eta}_i + 2\,\zeta\,\omega_i\dot{\eta}_i + \omega_i^2\eta_i = 0 \tag{6.1}$$

where ω is the modal frequency of ith mode, ζ is the damping ratio assumed to be constant for all modes $(0 < \zeta < 1)$ and η_i is the ith modal coordinate. The state space model for (6.1) is given by

$$\dot{x}_i = A_{ii}\, x_i \tag{6.2}$$

where

$$A_{ii} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \quad x_i^T = [\eta_i \quad \dot{\eta}_i] \tag{6.3}$$

Clearly the system is stable with eigenvalues $-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$.

We are now interested in obtaining perturbation bounds for the stability of the system (6.2) assuming variations in the matrix A_{ii}. If one considers this pertrubation as an 'unstructured' perturbation, which amounts to ignoring the structural information we have altogether, the elemental bound $\mu_{\varepsilon p}$ of (2.1) can be obtained. On the other hand, if one thinks of it as a 'structured' perturbation, then one can use the elemental bound $\mu_{\varepsilon Y}$ of (2.5) using U_n matrix. In fact one can do even better by not only considering the perturbation as 'structured' perturbation but also by exploiting the additional information that the variations can only be in frequency and damping ratio, thereby using the matrix U_e having the structure

$$U_e = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{6.4}$$

These three bounds, when specialized to the model of (6.3), can be given explicitly in terms of the modal data as follows:

The Lyapunov matrix equation is given by

$$A_{ii}^T P_i + P_i A_{ii} + 2I_2 = 0 \tag{6.5}$$

whose solution is given by

$$P_i = \begin{bmatrix} \left(\dfrac{1 + 4\zeta^2}{2\zeta\omega_i} + \dfrac{\omega_i}{2\zeta}\right) & 1/\omega_i^2 \\ \dfrac{1}{\omega_i^2} & \dfrac{1 + \omega_i^2}{2\zeta\omega_i^3} \end{bmatrix} \quad (6.6)$$

Note that in this case $P_{im} = P_i$ (i.e. all the elements of P_i are positive).

The bounds $\mu_{\varepsilon Pi}$ (for 'unstructured' perturbation), $\mu_{\varepsilon yi}$('structured' but with U_n) and μ_{yi} ('structured' with U_e of (6.4)) are given by

$$\mu_{\varepsilon Pi} = \frac{1}{2}\left[\frac{4\zeta\omega_i^3}{1 + \omega_i^4 + \omega_i^2(2 + 4\zeta^2) + \beta_1^{1/2}}\right] \quad (6.7a)$$

$$\mu_{\varepsilon Yi} = \frac{1}{2}\left[\frac{8\zeta\omega_i^3}{\omega_i^4 + \omega_i^2(2 + 4\zeta^2) + 4\zeta\omega_i + 1 + \beta_2^{1/2}}\right] \quad (6.7b)$$

$$\mu_{yi} = \frac{1}{2}\left[\frac{8\zeta\omega_i^3}{\omega_i^2 + 2\zeta\omega_i + 1 + \beta_3^{1/2}}\right] \quad (6.7c)$$

where

$$\beta_1 = 1 + \omega_i^8 + 8\,\omega_i^6\,\zeta^2 + \omega_i^4\,(16\zeta^4 - 2) + 8\,\omega_i^2\zeta^2$$

$$\beta_2 = [2\,\omega_i^8 + \omega_i^6\,(4 + 16\zeta^2) + 8\zeta\,\omega_i^5 + \omega_i^4\,(4 + 16\zeta^2 + 32\zeta^4) + \omega_i^3\,(16\zeta + 32\zeta^3) + \omega_i^2\,(4 + 16\zeta^2) + 8\zeta\omega_i + 2]\,\frac{1}{4\zeta^2\omega_i^6}$$

$$\beta_3 = 2\omega_i^4 + \omega_i^2\,(4 + 8\zeta^2) + 2$$

In figure 6, the variation of the bounds with respect to frequency ω (with constant damping ratio $\zeta = 0.001$) is presented. From this figure, it is clearly seen that both bounds $\mu_{\varepsilon yi}$ and μ_{yi} are always improved bounds over $\mu_{\varepsilon Pi}$

as expected. It is also interesting to note that μ_{yi} increases without bound as the frequency increases while $\mu_{\epsilon Yi}$ reaches a maximum (which depends on ζ) at a particular frequency and then decreases as ω is increased. This behavior may have some useful implications in synthesizing robust controllers. In fact this aspect is seen to have a useful extension to determine the trade off between the number of modes we control and the corresponding control gain one can use to achieve a given amount of regulation in the case where 'mode truncation' is treated as an additive perturbation and only control spillover occurs. This topic will be discussed in the next section.

Finally, the variation of ω_{max} (which results in the bound $\mu_{\epsilon Yi}$ with U_n) as the damping ratio ζ is varied in the range $0 < \zeta < 1$ is depicted in figure 7. From this figure it is interesting to note that ω_{max} decreases as the damping ratio ζ is increased. This aspect is to be investigated in future research.

6.2 Mode Truncation (Spillover Terms) as Perturbation

One design philosophy in LSS control is that of the 'Two-Model' theory of control design in which a higher order 'evaluation model' (which serves the role of the physical system in system evaluation) is driven by a lower order controller model. In this framework, the modeling error is, of course, 'mode truncation'. One prominent effect of mode truncation in the control design process is the instability caused due to the interaction with residual modes, labeled 'Spillover' [56]. There has been a considerable amount of work done on compensation for or elimination of 'spillover', notably by Balas [56], Sesak [57], Lin [58], Longman [59] among others. In this section, the unmodeled dynamic problem is treated as an additive perturbation and the time domain stability robustness analysis reported in the earlier sections is extended to the case of spillover instability. Using the explicit expressions for the

bounds obtained in the previous section, an algorithm is proposed, for the case of control spillover alone, by which one can determine, a priori the magnitude of the control gain in relation to the number of modes one wishes to control.

6.2.1 LSS Models and Mode Truncation as Perturbation

Consider the standard state space description of LSS evaluation model with N elastic modes:

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Dw \qquad x(0) = xo; \; x \to R^{n=2N}. \; u \to R^m \\
y &= Cx \qquad\qquad\qquad\qquad\qquad\qquad y \to R^k \\
z &= Mx + v \qquad\qquad\qquad\qquad\qquad z \to R^k
\end{aligned}
\tag{6.8a}
$$

where

$$x^T = [x_1^T, x_2^T, \ldots x_N^T]; \; x_i = \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} \tag{6.8b}$$

$$A = \text{Block diag. } [\ldots A_{ii} \ldots], \; A_{ii} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix} \tag{6.8c}$$

$$B^T = [B_1^T, B_2^T, \ldots B_N^T]; \quad B_i = \begin{bmatrix} 0 \\ b_i^T \end{bmatrix} \tag{6.8d}$$

$$D^T = [D_1^T, D_2^T, \ldots D_N^T]; \quad D_i = \begin{bmatrix} 0 \\ d_i^T \end{bmatrix} \tag{6.8e}$$

$$C = [C_1 C_2 \ldots C_N] \text{ and } M = [M_1, M_2, \ldots M_N] \tag{6.8f}$$

$$\xi \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} [w^T(t) \;\; v^T(\tau)] = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} (t-\tau) \tag{6.8g}$$

Let the above system be evaluated by the quadratic performance index

$$J = \lim_{t\to\infty} \frac{1}{t} \xi \int_o^t [y^T(\tau) Q y(\tau) + u^T(\tau) \rho_c R_o u(\tau)] d\tau \tag{6.9}$$

where scalar $\rho_c > 0$ and Q, R_o are (kxk) and (mxm) symmetric, positive definite matrices, respectively and ξ is the expectation operator.

Assuming the modes η_i are ordered in increasing order of frequency, let us retain the first N_r modes for control design purposes. Accordingly, the reduced order control design model of dimension $n_R(=2N_r) < n$ is given by

$$\begin{aligned}
\dot{x}_R &= A_R x_R + B_R u + D_R w \qquad x_R \rightarrow R^{\eta_R} \\
y_R &= C_R x_R \qquad\qquad (6.10) \\
z_R &= M_R x_R + v
\end{aligned}$$

where the above control design model is obtained by direct truncation of the full order model given by

$$\dot{x} = \begin{bmatrix} \dot{x}_R \\ \dot{x}_T \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & A_T \end{bmatrix} \begin{bmatrix} x_R \\ x_T \end{bmatrix} + \begin{bmatrix} B_R \\ B_T \end{bmatrix} u + \begin{bmatrix} D_R \\ D_T \end{bmatrix} w$$

$$(6.11)$$

$$y = [C_R \; C_T] \begin{bmatrix} x_R \\ x_T \end{bmatrix} \qquad z = [M_R \; M_T] \begin{bmatrix} x_R \\ x_T \end{bmatrix}$$

Let the full order optimal control for the reduced order model be given by

$$\begin{aligned}
u &= G_R \hat{x}_R \\
\dot{\hat{x}}_R &= A_R \hat{x}_R + B_R u + F_R(z - M_R \hat{x}_R) \qquad (6.12) \\
&= \hat{A}_R \hat{x}_R + F_R z \text{ where } \hat{A}_R = A_R + B_R G_R - F_R M_R
\end{aligned}$$

where G_R and F_R are the standard 'controller' and 'estimator' gain matrices respectively [19] such that the closed loop system matrix for the control design model given by

$$\bar{A}_R = \begin{bmatrix} A_R & B_R G_R \\ F_R M_R & \hat{A}_R \end{bmatrix} \tag{6.13}$$

is asymptotically stable under the usual assumptions of controllability and observability.

The closed loop system for the evaluation model is obtained by forcing the evaluation model with the controller of the control design model. Thus, we have

$$\begin{bmatrix} \dot{x}_R \\ \dot{\hat{x}}_T \\ \dot{x}_T \end{bmatrix} = \begin{bmatrix} A_R & B_R G_R & 0 \\ F_R M_R & \hat{A}_R & F_R M_T \\ 0 & B_T G_R & A_T \end{bmatrix} \begin{bmatrix} x_R \\ \hat{x}_R \\ x_T \end{bmatrix} + \begin{bmatrix} D_R & 0 \\ 0 & F_R \\ D_T & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{6.14a}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_R & 0 & C_T \\ 0 & G_R & 0 \end{bmatrix} [x_R \quad \hat{x}_R \quad x_T]^T$$

i.e., $\dot{X} = A_b X_b + D_b \bar{w}, \; y_b = G_b X_b$ (6.14b)

where the stability of the matrix is dictated by the spillover terms $B_T G_R$ and $F_R M_T$.

We now write $A_b = \bar{A}_1 + E_b$ where,

$$= \begin{bmatrix} A_R & B_R G_R & 0 \\ F_R M_R & \hat{A}_R & 0 \\ 0 & 0 & A_T \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & F_R M_T \\ 0 & B_T G_R & 0 \end{bmatrix} \tag{6.15}$$

i.e., the spillover terms are treated as an additive perturbation. Let us denote the matrix E as the 'spillover matrix'. Note that A_T, $\bar{A}_R$ and thus $\bar{A}_1$ are asymptotically stable matrices with A_T having a block diagonal structure.

Note that instability of the closed loop system may occur only if both the 'control spillover term $B_T G_R$' as well as the 'observation spillover term $F_R M_T$'

are present. If either only control spillover term alone or observation spillover term alone is present, then clearly the closed loop system is stable but significant performance degradation may occur. At the present stage of research, our analysis will be restricted to the case of 'control spillover' alone. Accordingly for this case the closed loop system matrix is written as

$$A_1 = \bar{A}_1 + E_1 \tag{6.16}$$

where

$$\bar{A}_1 = \begin{bmatrix} A_R & B_R G_R & 0 \\ 0 & A_R + B_R G_R & 0 \\ 0 & 0 & A_T \end{bmatrix} \tag{6.17a}$$

and

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B_T G_R & 0 \end{bmatrix} \tag{6.17b}$$

We now recall the expressions μ_{yi} and P_i of (6.7) & (6.6) respectively to be applied in our present analysis on mode truncation. From (6.6), (6.7) and the figure 6 the following design observation can be made.

Design Observation 6.1: An important property to note is that $\sigma_{max}(P_i U_e)_s$ decreases and thus μ_{yi} increases with increase in ω_i.

We now apply the above 'Perturbation Bound Analysis' to the case of matrix A_1 of (6.16). From (6.17) it can be seen that the maximum modulus element (in the worst case situation) of the 'spillover matrix' will be

$$\varepsilon_m = \bar{b}\; m\; \bar{g} \tag{6.18}$$

where

$$\bar{b} = |B_{ij}|_{max}, \quad \bar{g} = |G_{R_{ij}}|_{max}$$

(Note that $\bar{bm}$ can be easily determined for a given LSS model whereas $\bar{g}$ is not known before design).

Applying (2.6) to the matrix A_1 of (6.16), we conclude that A_1 (formed by forcing the evaluation model with a reduced order controller) is stable if

$$\varepsilon_m < \frac{1}{\sigma_{max}(P_m U_e)_s} \tag{6.19}$$

where P_1 satisfies $P_1\bar{A}_1 + \bar{A}_1^T P_1 + 2I_\alpha = 0$ $(\alpha = n + n_R)$ and

$$U_e = \begin{bmatrix} 0 & 0 \\ U_{eb2} & 0 \end{bmatrix} \text{ (because of the structure of E matrix of (6.17))}$$

and

$$U_{eb2} = \left[\begin{array}{c:cc} U_{e11} & U_{e12} & \cdots \\ \hline U_{e21} & U_{e22} & \cdots \\ \vdots & \vdots & \vdots \end{array}\right] \text{ where } U_{eij} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{6.20}$$

i.e., U_{eb2} contains $N_T = (N-N_r)$ number of U_{eij} submatrices because of the special structure of B matrix of (6.8). Note that

$$P_1 = \begin{bmatrix} P_R & 0 \\ 0 & P_T \end{bmatrix} \text{ where } P_R \text{ is } 2n_R x 2n_R \text{ matrix and}$$

$$P_T = \text{Block diag } [\ldots P_i \ldots] \quad i = 1,\ldots,N_T \tag{6.21}$$

and P_i is a 2x2 submatrix having the same structure of P_i as (6.6).

<u>Main Result</u>: Using the structural information of U_e, the design observation (6.1) and (6.7), it may be shown (after some matrix manipulation) that we can guarantee avoidance of instability due to spillover if

$$\bar{g} \text{ or } ||G_R|| < \frac{\mu_{yi}}{N\overline{bm}} < \frac{\mu_{yi}}{N_T\overline{bm}} \quad \text{for some given } i = 1,2,\ldots N \tag{6.22}$$

where μ_{yi} is given by (6.8) and N_T is the number modes truncated (=N-i).

Even though it is known that control spillover alone would not make the closed loop system (6.17) unstable, the result given in (6.22) is useful from a design point of view since this condition specifies apriori a bound on the control gain one can use for maintaining stability and its relationship to the number of modes one wishes to control.

Design Observation 6.2: Note that the bound on the control gain for (robust) stability increases with increase in the number of modes retained for control, which is consistent with the 'spillover' phenomenon.

We now apply the above result to the LSS example considered in [60].

Example 6.1 : The example considered is the Solar Telescope (SOT) model discussed in Yousuff, Skelton [60]. For our purposes we consider the model with the first eight elastic modes ordered in increasing order of frequency as the evaluation model. These modal frequencies are given by

Mode #	1	2	3	4	5	6	7	8
ω_i r/s	0.914	3.43	3.65	10.2	14.8	53.26	149.37	153.43

The matrix B (m=8) given in [60] is not reproduced here for lack of space. The damping ratio ζ = 0.001 is for all modes. For this model, the quantity $\overline{bm}$ of (6.18) is given by

$$\overline{bm} = 0.03124$$

The tabulation of μ_{yi} for i = 1,2,...N=8 (from equation (6.7c)) for each ω_i is as follows:

ω_i	μ_{yi}	$\mu_{yi}/N\bar{b}m$
0.914	0.689×10^{-3}	0.027×10^{-1}
3.63	0.559×10^{-2}	0.0223
3.65	0.563×10^{-2}	0.0225
10.82	0.178×10^{-1}	0.071
14.85	0.245×10^{-1}	0.098
53.86	0.892×10^{-1}	0.356
149.37	0.247	0.988
163.43	0.254	1.016

The entries in the last column suggest that the more modes we control, the higher the tolerable gain for stability.

The main advantage of the method is that the bound on the control gain depends only on the open loop data such as the modal frequency, mode shape slopes at actuator location, the damping ratio ζ, the number of control inputs and the number of modes included in the evaluation model.

Of course, the case which deals with combination of control and observation spillover is more useful and interesting and more research is needed to extend these concepts to this more complicated case. It is to be noted that even though ref. [68] treats this case, the main result stated is valid only for the case of control spillover alone and this restriction is inadvertently missing in the statement of the main result.

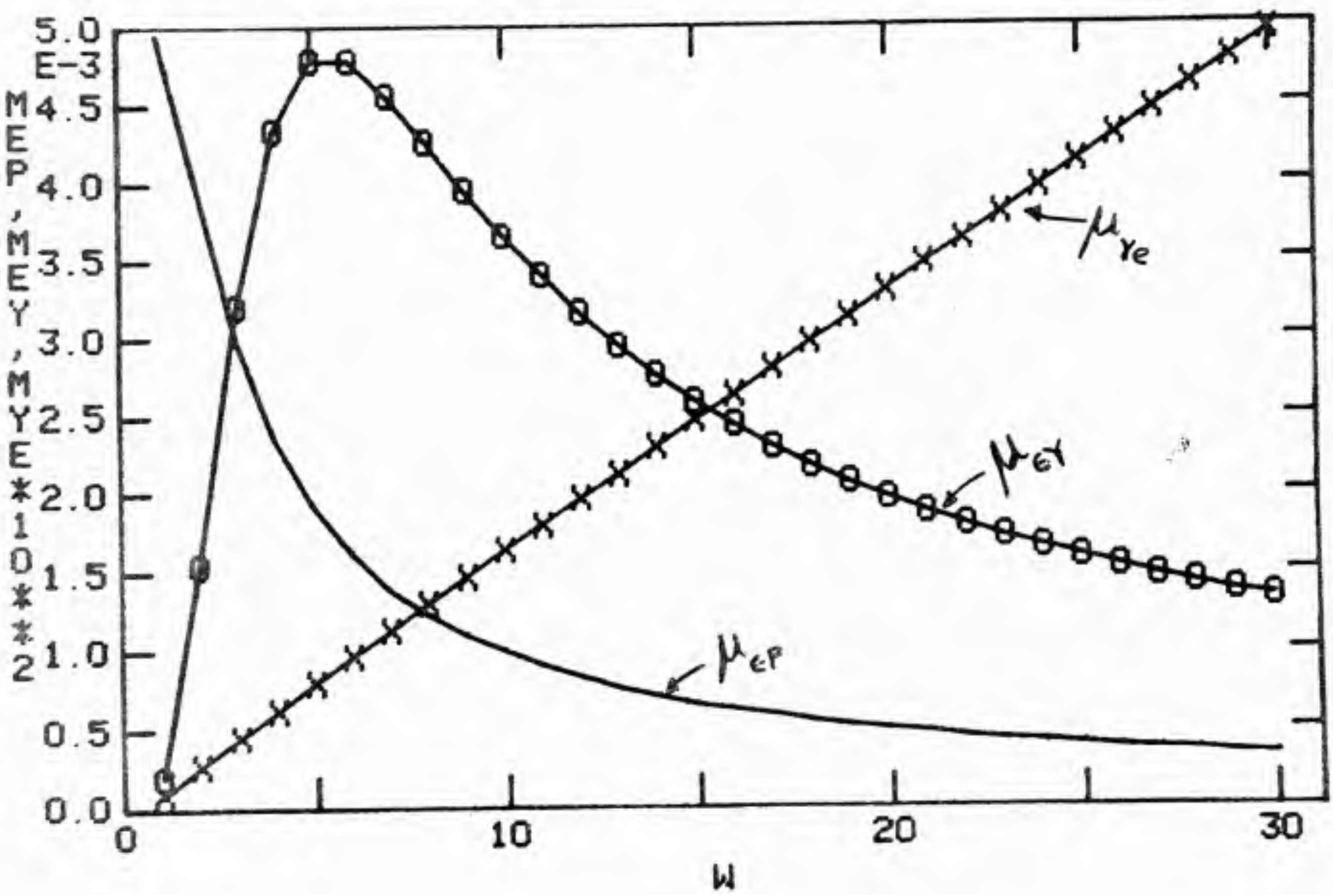


Figure 6: Variation of Bounds with frequency.

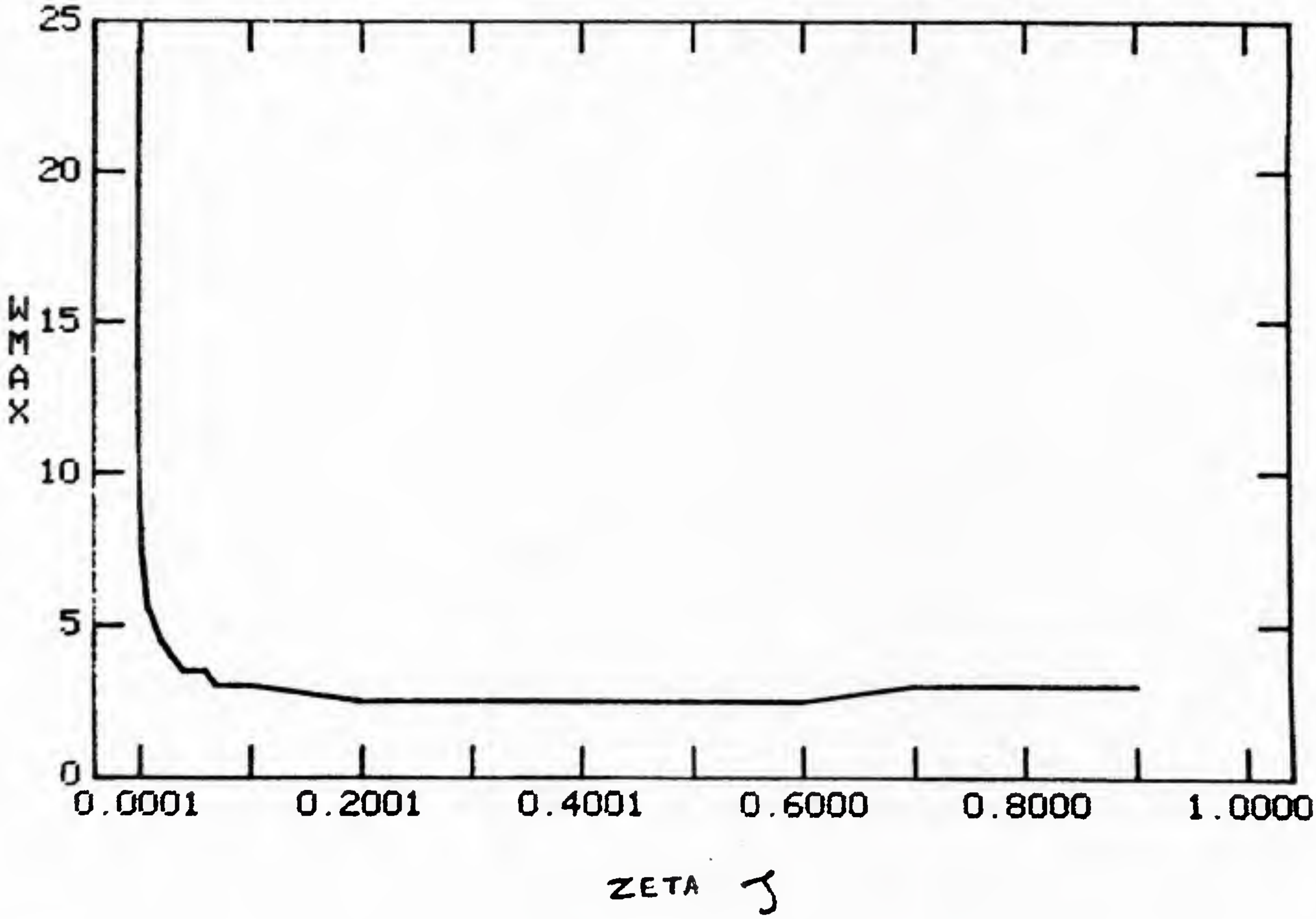


Figure 7: Variation of ω_{max} with damping ratio ζ.

VII. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

7.1 Work in Retrospect

The main theme of the research under the WPAFB contract has been to analyze and synthesize controllers for robust stability and performance for linear time invariant systems subject to linear time varying perturbations. First the aspect of stability robustness alone was considered. The main contribution of the research in this aspect is the development of perturbation bounds for robust stability for structured uncertainty. The proposed bound is such that it garners the structural information about the nominal as well as perturbation matrices into a unified expression. Another important contribution is to reduce the conservatism of the bounds. A solution is proposed to improve these bounds by employing a scaling transformation and the improvement of the proposed method over other existing methods is illustrated with the help of some realistic aircraft control examples. Next the application of these bounds in the analysis of stability of interval matrices is illustrated and the extension of reduction in conservatism of Lyapunov stability condition to ultimate boundedness control of linear systems is demonstrated convincingly. Finally a control design algorithm is presented to synthesize controllers for robust stability using the developed bounds.

Then the aspect of performance (regulation) robustness is addressed and perturbation bounds are again presented to guarantee prescribed degree of regulation for structured uncertainty. Another important contribution is to propose a design algorithm for linear regulators for robust regulation (which in turn guarantees stability) using the proposed 'Perturbation Bound Analysis'. The most significant feature of the proposed design methodology is that,

unlike some other methods in which the gain depends explicitly on the perturbation matrices, the proposed gain basically is a nominal gain but, of course, implicitly depends on the perturbation matrices as the robustness of the gain is investigated using the perturbation bounds. The advantages of this concept are that i) one need not have any restrictions placed on the perturbation matrices ΔA, ΔB (in fact one exploits the structure of ΔA, ΔB in the design procedure), ii) one need not invoke the assumption that the system be controllable, observable throughout the range of perturbations, and iii) the simplicity of the control law is maintained in the sense that the control law is still designed using the technique used for nominal design. The limitations of the method could be that for given perturbations it may not be possible to come up with a simple control law that accommodates the <u>entire</u> given range (however, it was possible to do so for a specific range that was considered in a realistic situation as demonstrated by the example in section 2.3). One way to circumvent this limitation is to be able to use more design variables and other methods to improve the bounds as much as we can.

Next, attention was focused on the application of these results to Large Space Structure models. Utilizing the special structure of these models and extending the results on the perturbation bounds with mode truncation as perturbation it was possible to determine the control gain bounds for maintaining stability in the case of control spillover alone and the relationship of these control gains to the number of modes one controls. This information will be extremely useful in analysis asd well as design situations of LSS control problems.

The publications listed as Refs. [61-70] are the result of this study. As it normally occurs, another result of this study is that many

interesting research topics surfaced for further investigation. These are presented in the next section.

7.2 Avenues for Further Research Which Need the Continued Support of the Air Force

1) The foremost area of research would be to further reduce the conservatism of the perturbation bounds.

2) It is important to develop necessary and sufficient conditions for the existence of a linear controller such that $\beta_{S.R.}$ (or $\beta_{P.R.}$) is positive for given perturbations.

3) An area of research would be to extend the development of explicit bounds for structured perturbation to time-invariant perturbations and examine the reduction in conservatism that can be achieved.

4) Another area of interest is to compare the proposed 'Perturbation Bound Analysis' approach to design with other relevant methods like the Guaranteed Cost Control of Chang and Peng [13] and the 'multimodel theory' of Ackermann.

5) It is also of interest to probe the relationship between the perturbation bound and the corresponding degree of stability measured by α, the real part of the dominant eigenvalue.

6) Another aspect for future research would be to extend the 'Perturbation Bound Analysis' for actuator-sensor location problems.

7) An area of extreme interest would be to use the perturbation bounds as a criterion in model/controller reduction and develop an algorithm for same and compare it with other relevant schemes.

8) One foremost area of research would be to extend the proposed analysis and design methodology to the case of combined modeling errors such as parameter variation, mode truncation and possibly nonlinearities.

9) There is need for probing into the comparison and contrast of frequency domain results and the proposed time domain results.

10) Of course, it is always instructive to apply the developed methodology to practical applications such as aircraft, spacecraft control problems and robotics.

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Appendix A

Proof of Theorem 2.1: Consider $\dot{x} = (A+E)\, x(t)$ (A1)

where $|E_{ij}|_{max} = \varepsilon$ (scalar) and $\Delta \equiv \varepsilon U_n$, where U_n is an nxn matrix with $U_{n_{ij}} = 1$ for all i, j=1,2,...,n.

Let $V(x) = x^T P\, x > 0$ be the Liapunov function for the system in (A1) where P is the symmetric positive definite solution of

$$A^T P + PA = -2I_n \tag{A2}$$

Then, $$\dot{V}(x) = -x^T\, 2I_n x + x^T (E^T P + PE) x \tag{A3}$$

Now

Let $\varepsilon < \dfrac{1}{\sigma_{max}(P_m U_n)_s}$

$\rightarrow \quad \sigma_{max}(P_m \Delta)_s < 1$

$\rightarrow \quad \sigma_{max}(PE)_s < 1$

$\rightarrow \quad |\lambda(PE)_s|_{max} < 1$

$\rightarrow \quad \lambda_i[(PE)_s - I_n] < 0$

$\rightarrow \quad [-I_n + (PE)_s]$ is negative definite

$\rightarrow \quad [-2I_n + E^T P + PE]$ is negative definite

$\rightarrow \quad \dot{V}(x)$ of (A3) is < 0 for all x

$\rightarrow \quad (A+E)$ of (A1) is stable

Appendix B

Proof of Theorem 2.3: Let A be stable and A_s be negative definite.

Let $\sigma_{max}(E) < \sigma_{min}(A_s)$

$\rightarrow \quad \sigma_{max}(E_s) < \sigma_{min}(A_s)$

$\rightarrow \quad |\lambda_i(E_s)| < -\lambda_{max}(A_s)$

$\rightarrow \quad \lambda_{max}(E_s) < -\lambda_{max}(A_s)$

$\rightarrow \quad \lambda_{max}(A_s + E_s) < 0$ (from Rayleigh's quotient formula for symmetric matrices, Noble, 1970).

$\rightarrow \quad A_s + E_s$ is negative definite

$\rightarrow \quad \text{Re}\lambda_i[(A + E)] < 0$

$\rightarrow \quad (A + E)$ is stable

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